RULES AND FORMULÆ

IN

ELEMENTARY MATHEMATICS.

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IN

ELEMENTARY MATHEMATICS

WITH NOTES.

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"The rules will help, if they be laboured and polished by practice."—BACON.

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PREFACE.

A Student, after carefully working out the different formulæ in Mathematics, still requires great familiarity with these in order readily to apply them to the solution of various problems. If he works out a good number of examples illustrating the formulæ, he gains this familiarity in the best possible manner. But immediately before his examination he ought to review all his previous work, and this will be most advantageously done, if he has the rules collected in a condensed form.

Experience has taught the Author that school boys want something more than this, that not only must they work numerous examples and have general papers on the back work, but that they must also have periodical repetition of the formulæ, which will otherwise soon become strangers to them, and be a constant barrier to their progress in the higher work.

He has, therefore, written the following pages to afford easy reference, and to have the rules and formulæ so arranged that their repetition with *vivâ voce* examination on their application may be more conveniently made a part of the class work.

The notes consist of propositions, which have been written at different times to explain and illustrate difficulties that have most frequently been encountered in teaching.

Care has been taken in writing out the formulæ to preserve the due order of the symbols, an arrangement that will be of great assistance to the memory, provided a like care is exercised by the learner. Any suggestions or corrections from Masters or Scholars will be thankfully received.

B. ARNETT.

ROYAL ACADEMY, GOSPORT,
June, 1873.

CONTENTS.

MATHEMATICAL FORMULÆ, NOTES, &c.

ERRATA.

p. 17, figure, interchange E and F.
p. 17, line 23, interchange E and F.

A million billions = one trillion.

Thus, the fourth figure is in the thousands' place,
the seventh..................millions' ....
the thirteenth...............billions' ....

DEF. Abstract numbers are those which have no reference to any particular kind of unit.

DEF. Concrete numbers are those which have reference to some particular kind of unit.

DEF. Every number divisible *only* by itself and unity is called a prime number.

DEF. When two or more numbers have no common measure but unity, they are said to be prime to each other.

All prime numbers, except 2 and 5, end in 1, 3, 7, or 9.

The prime numbers below 100 are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

To resolve a given number into its prime factors.

Divide it successively, and as often as possible by each of the prime numbers, beginning with the lowest prime divisor that will measure the given number; when the last quotient is prime, this prime quotient, and the several divisors, which have been used, are the prime factors required.

To find the G.C.M. *of two numbers.*

Divide the greater number by the less, divide the less by the remainder, divide that remainder by the next remainder, and so on, until the remainder is 0: the last divisor is the G.C.M.

To find the G.C.M. *of several numbers.*

Decompose them into their prime factors, and multiply together the *lowest* powers of those factors which are common to all.

To find the L.C.M. *of two numbers.*

Divide one of the given numbers by the G.C.M. of the two numbers, and multiply the other by this quotient.

To find the L.C.M. *of several numbers.*

Decompose them into their prime factors, and multiply together the *highest* powers of all the factors that occur.

DEF. A fraction is in its lowest terms when its numerator and denominator are prime to each other.

In simplifying expressions, when two quantities connected by the sign of multiplication are combined with

others by the sign *plus* or *minus*, these quantities must be first multiplied together, and the result then added to or subtracted from the other quantities.

To compare the values of vulgar fractions, reduce them to equivalent fractions with a common denominator.

Rule for Division of Decimals.

(*a*) Remove the decimal point in the *divisor* till it follows the extreme left-hand significant figure.

(*b*) Remove the decimal point in the *dividend* the same number of places.

The local value of the first figure in the *quotient* will be the same as that of the first significant digit on the left-hand of the *dividend*, provided that the divisor is contained in an equal number of figures of the dividend; if this is not the case, it will be of the value of the second significant digit.

To ascertain whether a vulgar fraction can be expressed by a terminating decimal.

Reduce the given fraction to its lowest terms,

and resolve its denominator into its prime factors;

if these prime factors be *only* 2 and 5, it can be expressed by a terminating decimal; otherwise, it cannot.

To find the value of a recurring decimal.

Subtract the integral number consisting of the non-recurring figures from the integral number, consisting of the non-recurring and recurring figures,

and divide by a number consisting of as many nines as there are recurring figures, followed by as many ciphers as there are non-recurring figures.

TABLES.

24 sheets *paper*	1 quire
20 quires	1 ream
10 reams	1 bale

12 dozen	1 gross

Length.

4 inches	1 hand
12 in.	1 ft.
3 ft.	1 yd.
6 ft.	1 fathom
$5\frac{1}{2}$ yds.	1 pole, rod, or perch
40 poles	1 furlong
8 fur.	1 mile
3 mls.	1 league
22 yds.	1 chain
100 links	1 chain
7·92 in.	1 link
1760 yds.	1 mile
80 chains	1 mile

Surface.

144 sq. in.	1 sq. ft.
9 sq. ft.	1 sq. yd.
$30\frac{1}{4}$ sq. yds.	1 sq. pole
40 sq. poles	1 rood
4 rds.	1 ac.
640 ac.	1 sq. mile
100000 sq. links	1 ac.
4840 sq. yds.	1 ac.
10 sq. chains	1 ac.

Volume.

1728 c. in.	1 c. ft.
27 c. ft.	1 c. yd.

1 cubic inch of pure water weighs 252·458 gr.; hence 1 c. ft. of pure water weighs 1000 oz. in Avoir. nearly.

Troy.

24 grains	1 dwt.
20 dwts.	1 oz.
12 oz.	1 lb.
5760 gr.	1 lb.

pure gold is 24 carats fine
standard......22 ....

Avoirdupois.

16 drams	1 oz.
16 oz.	1 lb.
14 lbs.	1 st.
28 lbs.	1 qr.
4 qrs.	1 cwt.
20 cwt.	1 ton
7000 gr. Troy	1 lb. Avoir.

Apothecaries'.

20 grains	1 scruple
3 sc.	1 dr.
8 drs.	1 oz.
12 oz.	1 lb.

the grain, ounce, and pound are the same as in Troy weight.

Capacity.

2 pints	1 quart
4 qts.	1 gallon
2 gals.	1 peck
4 pks.	1 bushel
8 bush.	1 qr.
5 qrs.	1 load
63 gals. *wine*	1 hhd.
2 hhds.	1 pipe
36 gals. *beer*	1 barrel
54 gals.	1 hhd.
2 hhds.	1 butt.

A gallon holds 10 lbs. Avoir. of pure water, or 70000 gr.; hence the volume of a gallon is 277·274 c. in.

Aliquot Parts

of £1.	
10*s*.	$\frac{1}{2}$
6*s*. 8*d*.	$\frac{1}{3}$
5*s*. 0*d*.	$\frac{1}{4}$
4*s*. 0*d*.	$\frac{1}{5}$
3*s*. 4*d*.	$\frac{1}{6}$
2*s*. 6*d*.	$\frac{1}{8}$
2*s*. 0*d*.	$\frac{1}{10}$
1*s*. 8*d*.	$\frac{1}{12}$
1*s*. 0*d*.	$\frac{1}{20}$

of 10*s*.	
5*s*. 0*d*.	$\frac{1}{2}$
3*s*. 4*d*.	$\frac{1}{3}$
2*s*. 6*d*.	$\frac{1}{4}$
2*s*. 0*d*.	$\frac{1}{5}$
1*s*. 8*d*.	$\frac{1}{6}$
1*s*. 3*d*.	$\frac{1}{8}$
1*s*. 0*d*.	$\frac{1}{10}$
10*d*.	$\frac{1}{12}$
6*d*.	$\frac{1}{20}$

of 6*s*. 8*d*.	
3*s*. 4*d*.	$\frac{1}{2}$
1*s*. 8*d*.	$\frac{1}{4}$
1*s*. 4*d*.	$\frac{1}{5}$
10*d*.	$\frac{1}{8}$
8*d*.	$\frac{1}{10}$
4*d*.	$\frac{1}{20}$

of 5*s*.	
2*s*. 6*d*.	$\frac{1}{2}$
1*s*. 8*d*.	$\frac{1}{3}$
1*s*. 3*d*.	$\frac{1}{4}$
1*s*. 0*d*.	$\frac{1}{5}$
10*d*.	$\frac{1}{6}$
7½*d*.	$\frac{1}{8}$
6*d*.	$\frac{1}{10}$
5*d*.	$\frac{1}{12}$
3*d*.	$\frac{1}{20}$

DIVISION.

In division, when both divisor and dividend are concrete numbers of the same kind, we reduce both numbers to the *same* denomination, and it must be carefully remembered that the remainder is of this same denomination.

Example.

A string 67 ft. 4 in. long is cut up into lengths of 6½ in. each; how many of these lengths are formed, and how many inches of string remain over?

```
in.          ft.  in.
6½           67   4
2            12
--          ---
13          808
              2
       ---------
   13) 1616 (124
       13
       --
        31
        26
        --
         56
         52
         --
          4
```

Here, before we could commence the division, we had to reduce both numbers to half-inches;

the number of lengths is 124, and the remainder is 4 half-inches, or 2 inches,

not 4 in., nor $\frac{4}{13}$ in., as gentlemen of different degrees of perverse ingenuity are apt to assert.

INTEREST.

DEF. Interest is the sum of money paid for the use of money borrowed for a certain time at a fixed rate per cent.

To find what principal at any given rate will amount to a certain sum in a given time.

Find the amount of £100 at the given rate in the given time, and state thus—

Amount of £100 : given amount
:: £100 : the principal required.

To find in what time a given principal will amount to a certain sum at any given rate.

By subtraction find the whole interest;

then find the interest on the given principal for *one* year, and state thus—

Interest for 1 yr. : whole interest
:: 1 yr. : number of yrs. required.

To find at what rate a given principal will amount to a given sum in a certain time.

By subtraction find the whole interest;

then find the interest on the principal for the given time at 1 per cent., and state thus—

Interest at 1 per cent. : whole interest :: £1 : Ans.

DISCOUNT.

DEF. The present worth of any debt due at some future time is such a sum that, if put out to interest at a given rate for the time during which the debt had to run, it would at the end of that time amount to the debt itself.

DEF. Discount is the simple interest of the present worth of the debt; and is equal to the debt *less* its present worth.

To find the present worth of a given sum.

Find the interest of £100 for the given time at the given rate per cent., and state thus—

£100 + its interest : £100

:: given sum : present worth required.

To find the discount, state—

£100 + its interest : this interest

:: given sum : discount required.

To find the debt when the discount, the time, and the rate per cent. are given.

Find the interest of £100 for the given time at the given rate per cent., and state thus—

Interest of £100 : given discount

:: amount of £100 : the debt required.

To find the time for which the debt has to run, when the debt, the discount, and the rate per cent. are given.

Find the present worth by subtracting the discount from the debt.

Now the discount is the simple interest of the present worth at the given rate for the unknown time.

Hence, find the interest of the present worth for 1 year at the given rate, and state thus—

Interest for 1 year : discount
:: 1 year : number of years required.

To find the rate per cent., when the debt, the discount, and the time are given.

Find the present worth by subtracting the discount from the debt.

Find the interest of the present worth at 1 per cent. for the given time, and state thus—

Interest at 1 per cent. : given discount :: £1 : Ans.

PROFIT AND LOSS.

I.

Prime cost : selling price :: 100 : 100 + gain per cent.

II.

Prime cost : selling price :: 100 : 100 – loss per cent.

STOCKS.

I.

In buying and selling stock we have—

Price of stock : amount of cash :: 100 : amount of stock.

II.

In selling out of one kind of stock, and buying into another, we have—

Price of 2nd. stock : price of 1st. stock
:: amount of 1st. stock : amount of 2nd. stock.

III.

The income that a man possesses by holding so much stock is obtained by multiplying the amount of stock by the rate per cent., and dividing by 100.

IV.

The income derived from the investment of so much cash in any stock at a certain price and rate per cent. is obtained by stating thus:—

Price of stock : amount of cash
:: rate per cent. : amount of income.

Be careful throughout to distinguish between cash and stock.

When brokerage is charged,
the price of the stock is increased to the buyer;
the price of the stock is diminished to the seller.

PROPORTIONAL PARTS.

Form fractions which shall have the numbers composing the given ratio as the respective numerators, and the sum of these numbers as the common denominator; take these fractional parts of the proposed quantity; they will be the parts required.

If we have given that one of two parts is a certain fraction of the other, the numbers representing the given ratio are the numerator and denominator; if the fraction is a decimal, then 1 and the decimal are the numbers composing the given ratio.

SQUARE ROOT.

To find the square root of a mixed number.

Reduce the number to an improper fraction, and take the square roots of the numerator and denominator.

If either the numerator or the denominator has *not* an exact square root, the fraction must be reduced to a decimal, and its approximate root be found.

To find the value of a fraction whose denominator is of the form $\sqrt{(a)} \pm \sqrt{(b)}$.

First multiply both numerator and denominator of the fraction by $\sqrt{(a)} \mp \sqrt{(b)}$, and then perform the operations indicated.

THE CHAIN RULE.

$$\begin{array}{lcl} \text{If } m_1 \text{ things of one kind} & = & n_2 \text{ things of a second,} \\ m_2 \text{ of the second} & = & n_3 \text{ of a third,} \\ m_3 \text{ of the third} & = & n_4 \text{ of a fourth,} \\ \&\text{c.} & = & \&\text{c.,} \\ m_p \text{ of the last kind} & = & n_1 \text{ of the first kind,} \end{array}$$

then $$m_1 m_2 m_3 \ldots m_p = n_2 n_3 n_4 \ldots n_p n_1.$$

THE USE OF COMMON LOGARITHMS.

DEF. The logarithm of a number to a given base is the index of the power to which the base must be raised to give the number.

The base of the common system is 10.

When a bar occurs over the first of the four figures in the logarithmic columns, it indicates that the common part consisting of three figures printed only in the first column must be increased by 1, before being placed in front of these four figures, or in front of any other four which follow in the same row.

Thus we must be careful, when finding a logarithm, to notice whether this bar occurs over the first of any four figures, which in the same row precede the set we are about to take.

The characteristic of the logarithm of a number is *one less* than the number of integral figures in the number.

When the number has no integral figures, the characteristic is negative and *one more* than the number of ciphers immediately following the decimal point.

The mantissæ are always positive.

The same mantissæ serve for the logarithms of all numbers, which have the same significant digits.

When only a certain number of decimal places is to be retained, strike off the rest of the figures, and add 1 to the last figure retained if the first of those struck off be 5 or greater than 5.

To divide a logarithm with a negative characteristic by any divisor, increase the negative characteristic, if necessary, so that it may be exactly divisible by the divisor, and make a proper compensation.

The logarithm of a product = the sum of the logarithms of the factors.

The logarithm of a quotient = the logarithm of the dividend *minus* the logarithm of the divisor.

To find the logarithm of a number raised to any power, integral or fractional, multiply the logarithm of the number by the index of the power.

To find the amount M of a given sum P in n years at compound interest,

$$M = PR^n,$$

where R is the amount of £1 in 1 year.

EUCLID.

A postulate is a claim stated by Euclid to the right of making an elementary construction, when needful, without any statement of the method employed.

An axiom is a statement the truth of which Euclid assumes without any demonstration of the same.

There are two kinds of propositions in Euclid, Problems and Theorems.

A problem is a proposition in which is determined the method by which lines, angles, or figures may be described, possessing specified properties.

A theorem is a proposition in which we demonstrate the truth of certain statements of the properties of specified lines, angles, or figures.

In most propositions there are five parts:

I. General Enunciation, in which there are always two statements, the statement of the datum or data, and the statement of what we are required to do or to prove.

II. Particular Enunciation.

III. Construction (drawing the figure).

IV. Demonstration.

V. Conclusion.

A corollary to a proposition is an inference which may be deduced immediately from that proposition.

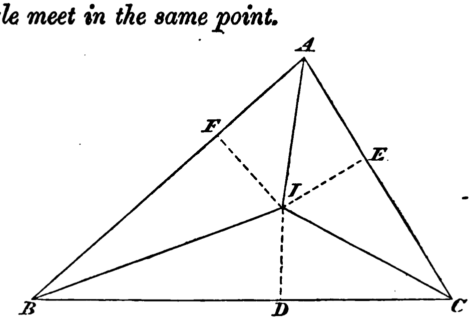
A lemma is a proposition which is demonstrated for the purpose of assisting us in the demonstration of another proposition.

We are said to prove a statement by the means of the "reductio ad absurdum" when we shew that the assumption of the falsity of the statement leads to conclusions that are known to be untrue.

In Geometrical *Synthesis* we demonstrate a new theorem, or solve a new problem, by successive steps of reasoning from results previously established.

In Geometrical *Analysis* we assume the truth of a theorem or the solution of a problem, and by reasoning from this assumption together with results previously established are *often* able to discover the steps, by which we may prove synthetically what has been assumed.

PROP. I. *The straight lines which bisect the angles of a triangle meet in the same point.*



Let ABC be any triangle.

Bisect the angles ABC, ACB by the straight lines BI, CI meeting at I.

Join AI, and from I draw ID, IE, IF perpendiculars to BC, CA, AB.

Then, by Euc. IV. 4,

$$ID = IE = IF.$$

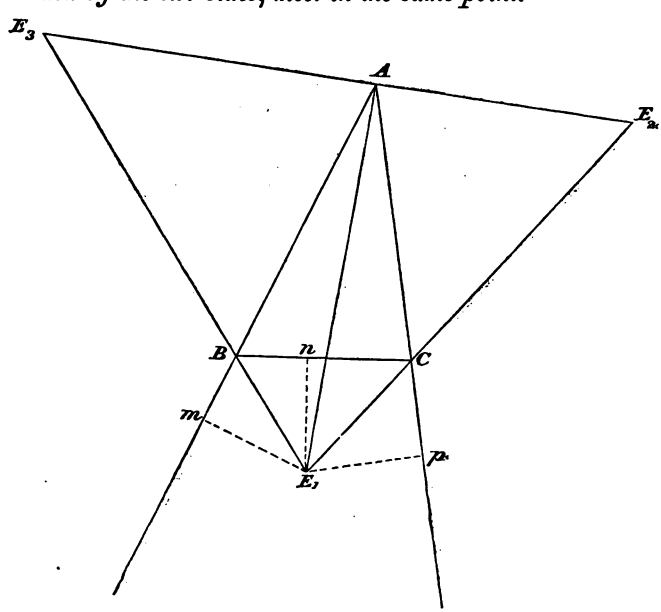
Because the angles at E and F are right angles, a circle can be described round the quadrilateral $AEIF$; and IE, IF cut off equal arcs of this circle (Euc. III. 28);

therefore the angles EAI, FAI, subtended by equal arcs, are equal (Euc. III. 27);

therefore IA bisects the angle A.

Thus the proposition is true.

PROP. II. *If two sides of a triangle be produced, the straight lines which bisect the two exterior angles thus formed, and the straight line which bisects the angle included by the two sides, meet in the same point.*



Let the sides AB, AC of the triangle ABC be produced, and let the exterior angles thus formed be bisected by the straight lines BE_1, CE_1, meeting at E_1.

Join E_1A, and draw E_1m, E_1n, E_1p perpendiculars to AB, BC, CA.

In the two triangles E_1Bm, E_1Bn,

the angle E_1Bm = the angle E_1Bn (constr.),

and the angle E_1mB = the angle E_1nB, each being a right angle;

also the side E_1B is common to the two triangles;

therefore $E_1m = E_1n$.

Similarly $E_1p = E_1n$;

therefore $E_1m = E_1n = E_1p$.

Again, because the angles at m and p are right angles,

a circle can be described round AmE_1p;

and F_1m, E_1p cut off equal arcs of this circle;

therefore, the angles E_1Am, E_1Ap, subtended by equal arcs, are equal;

therefore, AE_1 bisects the angle A.

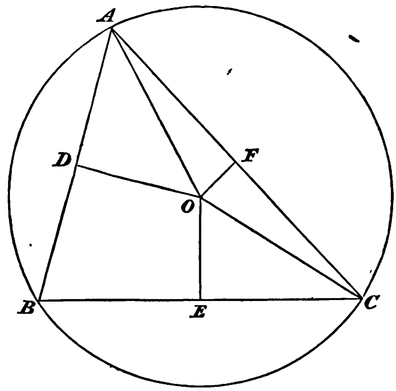
Thus the proposition is true.

Cor. Since $E_1m = E_1n = E_1p$, the circle described from the centre E_1 at the distance of any one of them will pass through the extremities of the other two, and touch the straight lines AB, BC, CA, because the angles at m, n, p are right angles.

This circle is called the *escribed* circle of the triangle with respect to the angle A.

Similarly we may find E_2, E_3, the centres of the escribed circles of the triangle with respect to the angles B and C.

Prop. III. *The straight lines drawn at right angles to the sides of a triangle through the middle points of the sides meet in the same point.*



Let D, E, F be the middle points of the sides AB, BC, CA, of the triangle ABC.

Draw DO, EO at right angles to AB, BC, meeting in the point O, and join OF.

Then, by Euc. IV. 5,
O is the centre of the circumscribed circle of the triangle;

therefore $$OA = OC.$$

Now $AF = FC$, and OF is common to the two triangles OFA, OFC;

also the base OA = the base OC;
therefore the angle OFA = the angle OFC;
therefore OF is at right angles to AC.
Thus the proposition is true.

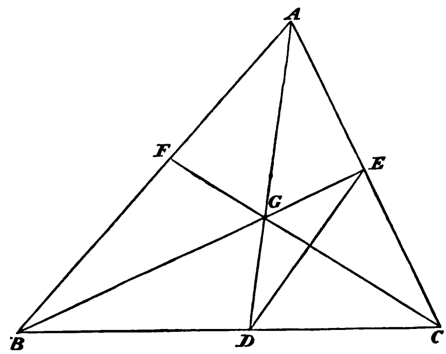
PROP. IV. *The straight lines drawn from the angles of a triangle to the middle points of the opposite sides meet in a point.*

Let D, E, F be the middle points of the sides BC, CA, AB of the triangle ABC, and join AD, BE meeting in the point G.

Join DE.

Then DE is parallel to AB (Euc. VI. 2).

Thus the triangles BAG, EDG are similar, as also the triangles EDC, ABC.



Therefore $BA : AG :: ED : DG$;

therefore $AG : DG :: BA : ED$ (Euc. v. 16)

$:: BC : DC$

$:: 2 : 1$;

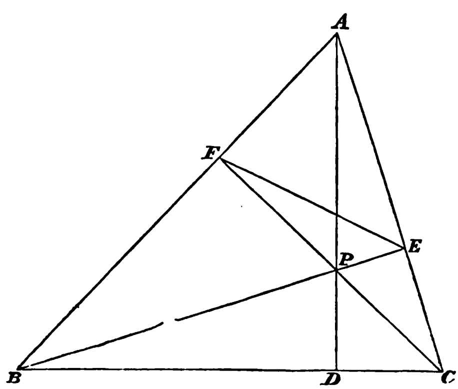
therefore $AG = 2DG$;

therefore $AG = \frac{2}{3}AD$.

In the same manner it may be shewn that CF meets AD in the point G, so that $AG = \frac{2}{3}AD$.

Therefore AD, BE, CF meet in the same point G. Q.E.D.

Prop. V. *The perpendiculars drawn from the angles of a triangle on the opposite sides meet in the same point.*



Let BE, CF be perpendiculars from the angles B and C of the triangle ABC on the opposite sides.

Let them meet at P; join AP, and produce it to meet BC at D.

First, let the triangle be not obtuse angled.

Join FE.

Then, since the angles AFP, AEP are right angles, a circle will go round $AEPF$.

Therefore the angle $EAP =$ the angle EFP (Euc. III. 21).

Again, because the angles BFC, BEC are right angles, a circle will go round $BFEC$.

Therefore the angle $EFC =$ the angle EBC (Euc. III. 21);

hence the angle $EAD =$ the angle EBD;

therefore a circle will go round $EABD$.

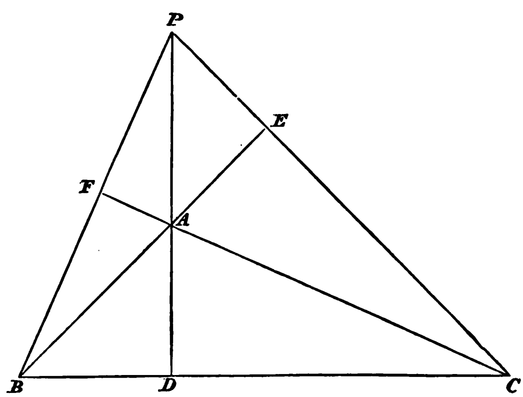
Therefore the angle $BDA =$ the angle BEA,

$= $ a right angle.

Therefore APD is perpendicular to BC.

Secondly, let A be an obtuse angle.

Then the perpendiculars BE, CF meet CA and BA produced in the points E, F.



Since BE, CF are perpendicular to the sides CP, BP of the triangle BPC;

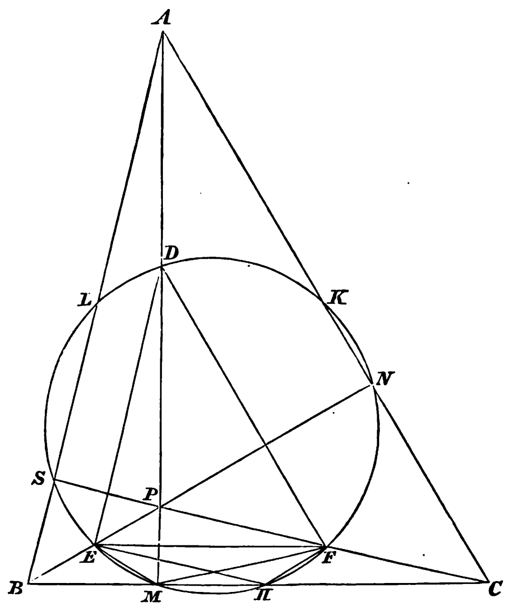
therefore, by the former case, PA is perpendicular to BC.

Thus the proposition is true.

NOTE. The point P in which the perpendiculars meet is called the orthocentre, or centre of perpendiculars.

THE NINE-POINT CIRCLE.

PROP. VI. *Let ABC be any triangle, P its orthocentre. The circle which passes through the middle points of PA, PB, PC will pass through the feet of the perpendiculars, and through the middle points of the sides of the triangle.*



Let D, E, F be the middle points of PA, PB, PC respectively,

M, N, S the feet of the perpendiculars,

H, K, L the middle points of the sides.

Join DE, EF, FD, EM, MF, EH, HF.

Then, because PMB is a right angle,

a semi-circle can be described round PMB, of which E the middle point of PB is the centre;

therefore $EP = EM$.

Similarly $FP = FM$;

and the base EF is common to the two triangles EPF, EMF;

therefore the angle EPF = the angle EMF.

But since the angles at N and S are right angles, a circle can be described round $PNAS$.

Therefore the angles EPF, BAC are together equal to two right angles.

Therefore the angles EMF, BAC are together equal to two right angles.

Now ED, DF are respectively parallel to BA, AC (Euc. VI. 2).

Therefore the angle EDF = the angle BAC.

Therefore the angles EDF, EMF are together equal to two right angles;

therefore M is on the circle passing through D, E, F.

Similarly, N and S are on the same circle.

Again, because $PFHE$ is a parallelogram (Euc. VI. 2),

therefore the angle EHF = the angle EPF = the angle EMF.

Therefore H is also on the circle through D, E, F.

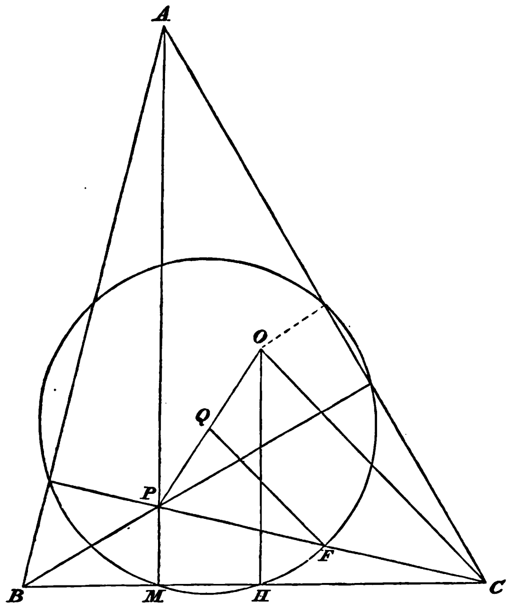
Similarly K and L are on the same circle.

Thus the nine points D, E, F, M, N, S, H, K, L are on the circumference of the same circle, which is called the nine-point circle of the triangle ABC.

PROP. VII. *To find the centre and radius of the nine-point circle.*

Let P be the orthocentre, and O the centre of the circumscribed circle of the triangle ABC; M, the foot of the perpendicular from A on BC; H, the middle point of BC.

Join OP, OH.



Then the centre of the nine-point circle is the middle point of OP.

For OH is perpendicular to BC, and therefore parallel to PM;

therefore the straight line which bisects MH at right angles must bisect OP,

and also pass through the centre of the nine-point circle (Euc. III. 1. Cor.).

Similarly, from the other sides of the triangle ABC two other straight lines can be obtained, which bisect OP, and pass through the centre of the nine-point circle;

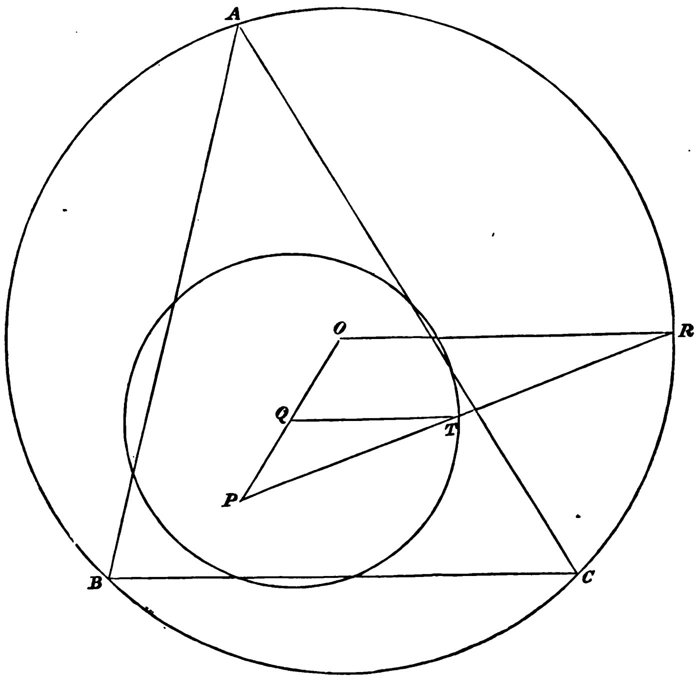
therefore Q, the middle point of OP, is that centre.

Let F be the middle point of PC, and join QF, OC.

Then QF is a radius of the nine-point circle,

and QF = half of OC = half the radius of the circumscribed circle.

PROP. VIII. *The middle point of every straight line drawn from the orthocentre to the circumscribing circle lies on the nine-point circle.*



From the orthocentre P let any straight line PR be drawn to meet the circumscribed circle at R, and let T be its middle point.

Join OR, QT.

Then PO is bisected at Q, and PR at T;

therefore QT is parallel to OR (Euc. VI. 2);

therefore TQP, ROP are similar triangles;

therefore $TQ : QP :: RO : PO$;

alternando, $TQ : RO :: QP : OP$

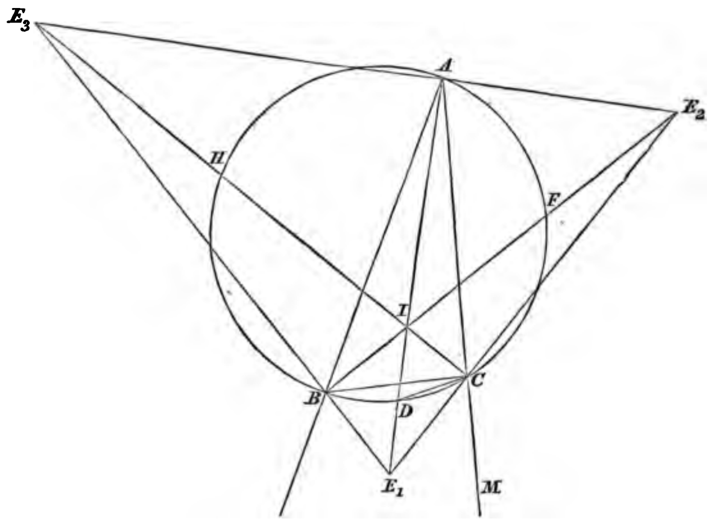
$:: 1 : 2$;

therefore $TQ = \frac{1}{2}RO =$ the radius of the nine-point circle.

Thus T is a point on the nine-point circle. Q.E.D.

PROP. IX. *The circumscribing circle of a triangle is the nine-point circle of the triangle, whose angular points are the centres of the escribed circles.*

Let ABC be any triangle; E_1, E_2, E_3 the centres of its escribed circles.



Join E_1C, CE_2, E_2A, AE_3, E_3B, BE_1.

Take I the centre of the inscribed circle (Euc. IV. 4).

Then AI, AE_1 both bisect the angle A;

therefore AIE_1 is a straight line.

Similarly BIE_2, CIE_3 are straight lines.

Again, because the angles ICB, BCE_1 are respectively halves of the angles ACB, BCM,

therefore ICE_1 is a right angle.

Similarly ICE_2 is a right angle;

therefore E_1CE_2 is a straight line, and E_3C perpendicular to E_1E_2.

Similarly E_2B, E_1A are respectively perpendicular to the straight lines E_3BE_1, E_2AE_3.

Therefore I is the orthocentre of the triangle $E_1E_2E_3$.

Let the circumscribed circle of the triangle ABC cut IE_1 in D, IE_2 in F,

and IE_3 in H; and join CD.

Then the angle $ADC =$ the angle ABC in the same segment;

therefore the angle ADC is double the angle IBC.

Now, since the angles IBE_1, ICE_1 are right angles,

a circle will go round ICE_1B, of which the centre is in IE_1;

and we have shewn that the angle IDC is double the angle IBC;

therefore D is the centre of this circle (Euc. III. 20).

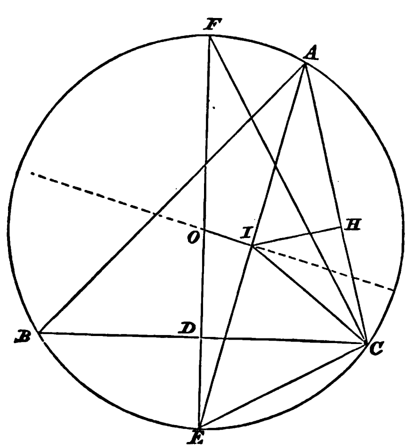
Thus the circumscribed circle of the triangle ABC passes through the middle point of IE_1.

Similarly it may be shewn to pass through the middle points of IE_2, IE_3;

therefore it is the nine-point circle of the triangle $E_1E_2E_3$.

PROP. X. *To find the distance between the centres of the inscribed and circumscribed circles of a triangle.*

Let ABC be a triangle, O the centre, and R the radius of the circumscribing circle.



From O draw OD perpendicular to BC, and produce it both ways to meet the circumference in E and F.

Then the arc BE = the arc EC;
therefore the angle BAE = the angle EAC;
therefore the centre of the inscribed circle is in AE.

Let I be the centre, and r the radius of the inscribed circle.

Join OI, IC, EC, FC, and draw IH perpendicular to AC.

The angle $$ICE = ICB + BCE$$
$$= \frac{C}{2} + \frac{A}{2}$$
$$= EIC \text{ (Euc. I. 32)};$$
therefore $$EI = EC.$$

Now the angle IAH = the angle EFC (Euc. III. 21),
and the right angle $IHA = ECF$ in a semi-circle;
therefore the triangles EFC and IAH are similar;

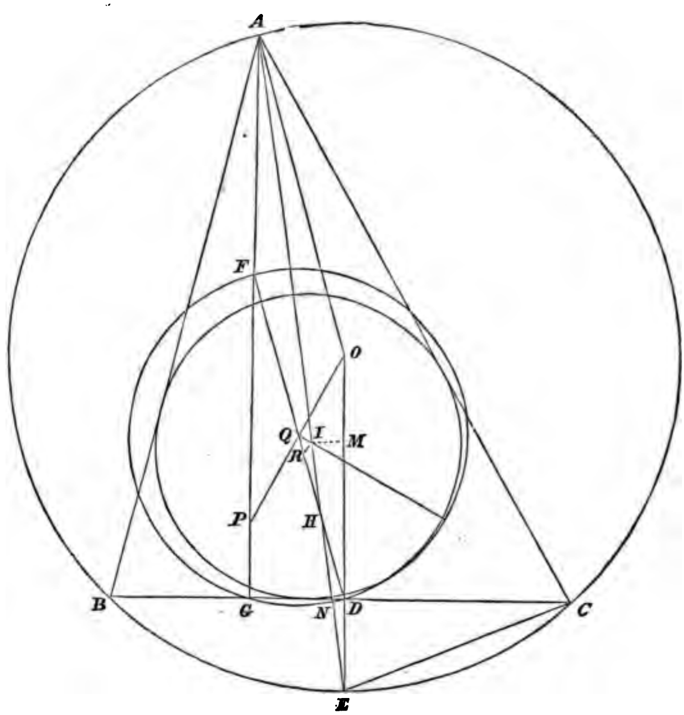
therefore $$EF : EC :: IA : IH \text{ (Euc. VI. 4)},$$
that is $$2R : EI :: IA : r;$$
therefore $$EI.IA = 2Rr,$$
but $$EI.IA = (R + OI)(R - OI) \text{ (Euc. III. 35)}$$
$$= R^2 - OI^2 \text{ (Euc. II. 5)};$$
therefore $$OI^2 = R^2 - 2Rr.$$

PROP. XI. *The inscribed circle of a triangle touches the nine-point circle of the same triangle.*

Let ABC be a triangle.

Let P be the orthocentre, I the centre of the inscribed circle, O the centre of the circumscribed circle, Q the centre of the nine-point circle.

From O draw OD perpendicular to BC, and produce it to meet the circumference of the circumscribed circle in E.



Let F be the middle point of AP, and join DF.

Then because DGF is a right angle,

and D, F are on the nine-point circle;

therefore DF passes through the centre Q.

Thus the triangles FQP, DQO are equal in all respects (Euc. I. 4);

therefore $$AF = PF = OD;$$

therefore $AFDO$ is a parallelogram ..... (i).

Join AE passing through I, and intersecting DQ in H.

Then by (i) DH is parallel to OA;

therefore $DH = DE$ (Euc. VI. 4);

also by the similar triangles EDC, ECA,

$$ED : EC :: EC : EA;$$

therefore

$$EI^2 = EC^2 = ED.EA = 4ED.DQ \text{ (Prop. VII.) .... (ii).}$$

Join QI, and draw IM perpendicular to OD, IR perpendicular to DQ, and DN perpendicular to EI.

Thus we have the four similar triangles EMI, HRI, HND, END;

therefore $EM : EI :: HR : IH :: EH : 2ED$ ... (iii).

Now $QI^2 = IH^2 + QH^2 - 2HR.HQ$ (Euc. II. 13),

and $IH^2 = (EI - EH)^2 = \left(EI - \dfrac{2DE.EM}{EI}\right)^2$ from (iii)

$$= EI^2 - 4DE.EM + 4DE^2.\frac{EM^2}{EI^2},$$

$$HQ^2 = (DQ - DE)^2,$$

$$2HQ.HR = 2(DQ - DE).\frac{EM}{EI}\left(EI - \frac{2DE.EM}{EI}\right)$$

$$= 2EM(DQ - DE) - 4\frac{EM^2}{EI^2}(DQ - DE)DE$$

$$= 2EM.DQ - 2DE.EM$$

$$+ 4DE^2.\frac{EM^2}{EI^2} - EM^2.\frac{4ED.DQ}{EI^2};$$

therefore $QI^2 = EI^2 - 2DE.EM + (DQ - DE)^2$

$$- 2EM.DQ + EM^2 \text{ from (ii)}$$

$$= (DQ + DE)^2 - 2EM(DQ + DE) + EM^2$$

$$= (DQ + DE - EM)^2$$

$$= (DQ - DM)^2;$$

therefore $QI = DQ - DM$,

or $DQ = QI + DM$,

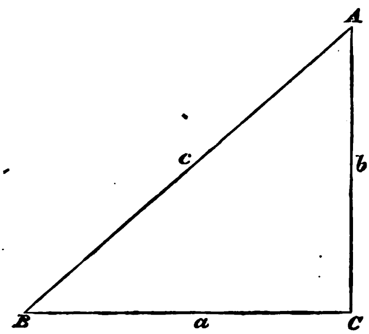
but DQ is the radius of the nine-point circle, and $DM =$ the radius of the inscribed circle;

thus the proposition is true.

MENSURATION.

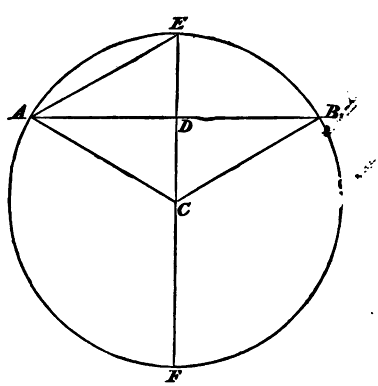
N.B. Before using any rule in Mensuration, it is necessary to express the given lengths in the same denomination.

In any right-angled triangle,



$$c^2 = a^2 + b^2.$$

Let AB be any chord of a circle, FE a diameter at right angles to it, AE the chord of half the arc AEB, DE the height, and l the length of the arc AEB.



(i) $ED \times EF = EA \times EA.$

(ii) $ED \times DF = AD \times DB.$

(iii) $360^\circ : \angle ACB :: C : l.$

(iv) $C = 2\pi r,$

DEFINITIONS.

A trapezoid is a four-sided figure, of which two opposite sides are parallel.

A zone of a circle is the figure contained by two parallel chords, and the parts of the circumference they cut off.

A lune is a plane figure formed by two segments of circles which have a common chord, and are on the same side of it.

A prism is called *right* when the parallelograms between the ends are rectangles, and *oblique* when they are not.

A frustum of a solid is a slice of it contained between the base, and any plane parallel to the base.

A wedge is a solid bounded by five planes; the base is a rectangle; the two ends are triangles; and the other two faces are trapezoids.

The trapezoids intersect in the edge of the wedge.

A prismoid is a solid, having for its ends any two parallel plane rectilineal figures of the same number of sides, and having all its faces trapezoids.

A zone of a sphere is the part of the sphere contained between two parallel planes; the height of the zone is the perpendicular distance between the two parallel planes.

A solid ring is a solid produced by turning a circle round any straight line, which is in the same plane as the circle, but does not cut it.

The length of the ring is the circumference of the circle, which passes through the centres of all the circular or transverse sections.

AREAS.

Let s = half the sum of the sides of any figure.

The area of

a rectangle = length × breadth;

the walls of a room = sum of length and breadth of of the room × its height × 2;

a parallelogram = base × height;

a trapezoid = product of half the sum of the parallel sides and the perpendicular distance between them;

a triangle = $\frac{1}{2}$ (base × height)

$= \sqrt{\{s(s-a)(s-b)(s-c)\}}$;

a quadrilateral = $\frac{1}{2}$ (diagonal × sum of perpendiculars);

any polygon = sum of the areas of the triangles composing it;

a quadrilateral inscribed in a circle $= \sqrt{\{(s-a)(s-b)(s-c)(s-d)\}}$;

a circle $= \pi r^2$;

a circular ring $= \pi(r_1^2 - r_2^2) = \pi(r_1 + r_2)(r_1 - r_2)$;

a sector of a circle $= \frac{1}{2}lr$, where l = the length of the arc.

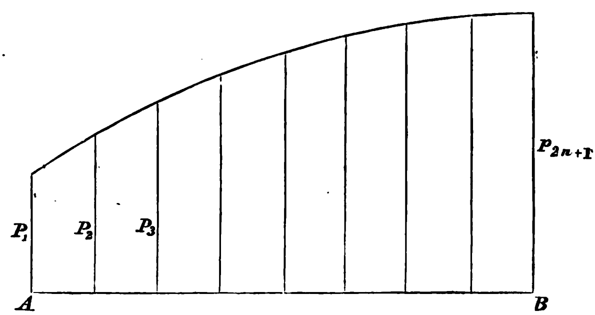
To find the area of a segment of a circle, subtract the area of the triangle formed by the radii and the chord from the area of the sector with the same arc.

To find the area of a zone of a circle, subtract the area of the segment whose base is the lesser chord from that of the segment, whose base is the greater chord.

If, however, the chords are on opposite sides of the parallel diameter, add the areas of the two zones formed by the diameter and each of the chords.

The areas of similar figures are to each other as the squares of homologous sides.

Simpson's rule for finding the approximate area of any figure bounded by a straight line, a curve, and two ordinates.



Divide AB into an even number ($2n$) of equal parts, each of which $= h$, and draw the ordinates $p_1, p_2, p_3, \dots p_{2n+1}$.

Then the area

$$=\frac{h}{3}\{p_1+p_{2n+1}+2(p_3+p_5+\dots+p_{2n-1})+4(p_2+p_4+\dots+p_{2n})\}.$$

CURVED SURFACES.

Let h be the altitude of a solid.

The area of the curved surface of a right circular cylinder $=2\pi rh$.

The area of the surface of a solid ring $=2\pi rl$.

The area of the curved surface of a right circular cone $=\pi r\sqrt{(r^2+h^2)}$.

The area of the curved surface of a frustum of a right circular cone $=\pi(r_1+r_2)\times$ slant height of frustum.

The area of the surface of a sphere $=4\pi r^2$.

The area of the curved surface of a zone, or of a segment of a sphere $=2\pi rh$, where r is the radius of the sphere.

VOLUMES.

Let A be the area of the base of any solid.

The volume of

a parallelepiped,
a prism,
or a cylinder $= Ah$;

a pyramid,
or a cone $= \dfrac{Ah}{3}$;

a frustum of a pyramid, or of a cone $= \dfrac{\pi h}{3}(r_1^2 + r_2^2 + r_1 r_2)$, where r_1, r_2 are the radii of the two ends;

a solid ring $= \pi r^2 l$;

a sphere $= \frac{4}{3}\pi r^3$;

a segment of a sphere $= \dfrac{\pi h}{6}(3r^2 + h^2)$, where r is the radius of the base of the segment;

a zone of a sphere $= \dfrac{\pi h}{6}\{3(r_1^2 + r_2^2) + h^2\}$.

To find the volume of a wedge.

Add the length of the edge to twice the length of the base; multiply the sum by the width of the base, and the product by the height of the wedge; one-sixth of the result will be the volume.

To find the volume of a prismoid.

Add together the areas of the two ends and four times the area of a section parallel to the two ends and midway between them; multiply the sum by the height and divide the product by six.

ALGEBRAICAL SYMBOLS OF RELATION.

The equality of two quantities is expressed by the sign $=$; thus $a=b$ signifies that a is equal to b.

When two quantities are unequal it is expressed by the sign $>$ or $<$, which is placed between the two quantities, care being taken to turn the *point* of the sign towards the *lesser* quantity.

Thus $a>b$ and $b<a$ respectively signify that a is greater than b, and that b is less than a.

Hence we use the notation $> <$ to signify *is greater or less than.*

The symbol $\therefore$ signifies then or therefore;
the symbol $\because$ signifies since or because.

An algebraical expression, called also a formula, is any combination of algebraical symbols.

A simple expression is one which does not contain either of the signs $+$ and $-$; thus a^2, $7a^3b^2c$, $\dfrac{3ab}{5c^3d}$ are simple expressions.

When two or more simple expressions are connected by the signs $+$ and $-$, they together constitute a compound expression; thus $a+b-c$, $5a^2b^3+a^3b^2-4a^5b+c^4n$ are compound expressions.

The simple expressions which compose a compound expression are called its terms.

A binomial is an expression of two terms.

A trinomial is an expression of three terms.

A multinomial or polynomial is an expression of more than three terms.

The sign of multiplication is $\times$ or $.$, which is placed between two numbers to be multiplied together; thus $a\times b$ or $a.b$ signifies that a is to be multiplied by b, and is read a into b. We very often suppress the sign

of multiplication; thus the product of a and b is written simply ab; in the same way $3a$ is the product of 3 and a, but when both the numbers are represented by figures, we must always use the sign $\times$; thus 3×5, not 3.5, nor 35.

N.B. When the expressions whose product we wish to indicate are compound, they must be enclosed by brackets; thus the product of $a+b+c$ and $d+e$ is written $(a+b+c)(d+e)$.

A vinculum, which is a line drawn over the quantities to be taken collectively, serves the same purpose as brackets; thus the above product may be written $\overline{a+b+c}\,.\,\overline{d+e}$.

It is a serious mistake to omit writing either of these signs; suppose them omitted in the above product, and we have $a+b+c \times d+e$, which denotes that c is multiplied into d, and that a, b, and e are added to the result; but this is evidently very different from the result of multiplying the sum of a, b, c by the sum of d and e, which is indicated by the symbols $(a+b+c)(d+e)$.

DEF. An algebraical product is of n dimensions when the sum of the indices of the quantities composing the product is n.

DEF. An algebraical expression is homogeneous when all its terms are of the same dimensions.

It is useful to notice this property of expressions, for if we perform any operations with homogeneous expressions, they will always remain homogeneous, and thus afford some test of the accuracy of our work.

FACTORS.

One of the most useful results in mathematics is that

$$a^2 - b^2 = (a+b)(a-b),$$

or, expressing it in words,

sum into difference = difference of squares.

Hence, if we have an expression of two terms with the negative sign between them, we can resolve it into factors thus:

Take the square root of each term;
then, the sum of the roots will be one factor,
the difference of the roots will be the other factor.

$$\begin{aligned} &x^4 + 4y^4 \\ &= (x^2 + 2y^2)^2 - 4x^2y^2 \\ &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2). \end{aligned}$$

We have by multiplication

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

Hence, *to resolve an expression of the latter form into factors*, we must find two numbers, whose algebraical sum is the coefficient of x, and whose product is the third term.

Thus, for example,

$$\begin{aligned} x^2 + 9x + 20 &= (x + 4)(x + 5), \\ x^2 - 9x + 20 &= (x - 4)(x - 5), \\ x^2 + x - 20 &= (x - 4)(x + 5), \\ x^2 - x - 20 &= (x + 4)(x - 5). \end{aligned}$$

We see then that,

(a) when the third term is positive,
the second terms of the factors are both positive,
or both negative;
they are both positive, when the coefficient of x is positive;
they are both negative, when the coefficient of x is negative:

(β) when the third term is negative,

one of the second terms of the factors is positive,

and the other negative;

the positive one is the greater, when the coefficient of x is positive;

the negative one is the greater, when the coefficient of x is negative.

$$x^2+y^2=\{x+\sqrt{(-1)}\,y\}\,\{x-\sqrt{(-1)}\,y\}.$$

$$x^3+y^3=(x+y)\,(x^2-xy+y^2).$$

$$x^3-y^3=(x-y)\,(x^2+xy+y^2).$$

$$x^4-y^4=(x-y)\,(x^3+x^2y+xy^2+y^3).$$

$$x^4-y^4=(x+y)\,(x^3-x^2y+xy^2-y^3).$$

$$\frac{x^n-1}{x-1}=x^{n-1}+x^{n-2}+\ldots+x^2+x+1.$$

In resolving expressions of the preceding forms into factors, we may verify our results by the following test.

If a is a number such that, being substituted for x, it renders any expression containing x equal to zero, then $x-a$ is a factor of the expression.

If n be an integer,

x^n+y^n is divisible by $x+y$, when n is odd,

by $x-y$ never;

x^n-y^n is divisible by $x+y$, when n is even,

by $x-y$ always.

When the divisor is $x-y$, all the terms of the quotient are positive.

When the divisor is $x+y$, the terms of the quotient are alternately positive and negative.

$$(x+y)^2=x^2+y^2+2xy.$$

$$(x-y)^2=x^2+y^2-2xy.$$

$$\left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2.$$

$$\left(x-\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}-2.$$

$$(x+y)^3=x^3+y^3+3xy\,(x+y).$$

$$(x-y)^3=x^3-y^3-3xy\,(x-y).$$

$$x^3+y^3+z^3-3xyz=(x+y+z)\,(x^2+y^2+z^2-xy-yz-zx).$$

DEF. The Greatest Common Measure or the Highest Common Factor of two or more algebraical expressions is the expression of *highest dimensions* with respect to some one letter of reference, which will divide each of the former without remainder.

In Long Division, and in finding the G.C.M., arrange the expressions in the order of magnitude of the indices of the letter of reference.

If at any stage in the process of finding the G.C.M., we divide or multiply the remainder by any expression, care must be taken to notice whether this is a factor of the preceding divisor, or contains a factor, which is also a factor of the preceding divisor.

To reduce a fraction to its lowest terms.

Resolve both numerator and denominator into factors, and remove those common to both; but if this cannot be done by inspection, divide both numerator and denominator by their G.C.M.

DEF. The Least Common Multiple of two or more algebraical expressions is the expression of *lowest dimensions*, which is exactly divisible by each of them.

If $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \&c.,$$

then each fraction $$= \frac{a_1 + a_2 + a_3 + \dots}{b_1 + b_2 + b_3 + \dots}$$

$$= \frac{pa_1 + qa_2 + ra_3 + \dots}{pb_1 + qb_2 + rb_3 + \dots}$$

$$= \sqrt[n]{\left(\frac{pa_1^n + qa_2^n + ra_3^n + \dots}{pb_1^n + qb_2^n + rb_3^n + \dots}\right)}$$

$$\text{in fine} = \frac{f(a_1, a_2, a_3, \dots a_n)}{f(b_1, b_2, b_3, \dots b_n)},$$

f being any homogeneous function of the first degree.

Any given case of this theorem can be proved by supposing

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \&c. = k,$$

so that b_1k, b_2k, ... may be substituted for a_1, a_2....

$$a^m \times a^n = a^{m+n}.$$

$$(a^m)^n = a^{mn}.$$

$$a^n \times b^n = (ab)^n.$$

$$a^0 = 1.$$

$$a^{-n} = \frac{1}{a^n}.$$

The squares of all numbers, negative as well as positive, are positive.

An algebraical expression is said to be real, when it does not involve an even root of a negative quantity; otherwise, it is said to be imaginary or impossible.

If then an expression involves even roots, the condition that it should be real is, that the quantities under these root signs should be positive.

To rationalise a binomial surd of the form $\sqrt{a} \pm \sqrt{b}$.
Multiply by $\sqrt{a} \mp \sqrt{b}$.

If $$a + \sqrt{b} = c + \sqrt{d},$$
then $$a = c, \text{ and } b = d.$$

If $$\frac{a}{b} = \frac{c}{d},$$
then $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

EQUATIONS.

In equations of the form,

$$\frac{x \pm a}{x \pm b} \pm \frac{x \pm c}{x \pm d} = \frac{x \pm e}{x \pm f} \pm \frac{x \pm g}{x \pm h},$$

express each fraction as a mixed quantity; thus, taking the upper signs,

$$\text{the first fraction} = 1 + \frac{a-b}{x+b}.$$

An equation may be cleared of a surd by transposing the terms so that the surd shall form one side, and the rational quantities the other, and then raising both sides to that power which will rationalise the surd. But we must always verify the results obtained, for we are not sure that they will satisfy the equation; they *may* satisfy an equation, which differs from the original one in the sign of the surd.

To solve the simultaneous equations,

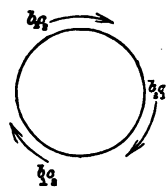
$$a_1x + b_1y + c_1z = d_1 \ldots\ldots\ldots\ldots\ldots\text{(i)},$$
$$a_2x + b_2y + c_2z = d_2 \ldots\ldots\ldots\ldots\ldots\text{(ii)},$$
$$a_3x + b_3y + c_3z = d_3 \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Multiply (i) by $b_2c_3 - b_3c_2$,

(ii) by $b_3c_1 - b_1c_3$,

(iii) by $b_1c_2 - b_2c_1$.

We must be careful in remembering this and other similar formulæ to take the subscripts in the same order of succession.



Thus we pass from b_2 to b_3, from b_3 to b_1, and from b_1 to b_2, as indicated in the figure.

In quadratics, before completing the square, remember that the coefficient of the square of the unknown symbol must be $+1$.

If α and β be the roots of the equation

$$ax^2 + bx + c = 0,$$

we have
$$\alpha + \beta = -\frac{b}{a};$$

$$\alpha\beta = \frac{c}{a};$$

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

α and β are equal, but of opposite sign, when $b = 0$;

real and different	$b^2 > 4ac$;
real and equal	$b^2 = 4ac$;
imaginary	$b^2 < 4ac$.

If both sides of an equation can be divided by an expression containing the unknown, divide by it, and observe that this expression being made equal to zero gives one or more values of the unknown.

When equations in x and y are homogeneous, and of the second degree, put $y = mx$.

PROBLEMS.

In problems involving sums of money, weights, and measures, &c., always begin by reducing the quantities to the same denomination.

If a man, a running tap, &c., can perform a certain work in x hours,

$\frac{1}{x}$ will represent the part of the work done in one hour.

In problems involving rates of walking, running, &c.,

$$\text{distance} = \text{rate} \times \text{time}.$$

In problems in which it is required to find the place of meeting, equate the times.

In problems in which it is required to find the time of overtaking, equate the distances.

A number whose digits are x and $y = 10x + y$.

In clock problems if x be the number of minute divisions the minute hand has passed over,

then $\frac{x}{12}$ = the number of divisions the hour hand has passed over in the same time.

If a man buy x articles for y pence,

$$\text{each costs } \frac{y}{x} \text{ pence,}$$

$$\text{and } \frac{x}{y} = \text{the number he gets for one penny.}$$

We cannot express the square root of a binomial in a rational form.

For example, the square root of $a^2 + b^2 = \sqrt{(a^2 + b^2)}$,

and not $a + b$;

the square root of $x^4 - a^2x^2 = \sqrt{(x^4 - a^2x^2)}$, or $x\sqrt{(x^2 - a^2)}$,

and not $x^2 - ax$.

RATIO, PROPORTION, AND VARIATION.

Ratio is the relation, which one quantity bears to another of the same kind with respect to magnitude, the comparison being made by considering what multiple or part the first is of the second.

The ratio of a to b is expressed by the fraction $\frac{a}{b}$, or thus, $a : b$.

In the ratio, $a : b$, $\begin{cases} a \text{ is the antecedent,} \\ b \text{ is the consequent.} \end{cases}$

Ratios are compounded by multiplying together the fractions by which they are denoted.

The duplicate ratio of $a : b$ is $a^2 : b^2$.

The triplicate ratio of $a : b$ is $a^3 : b^3$.

Four quantities, a, b, c, d are proportionals, when

$$\frac{a}{b} = \frac{c}{d}.$$

Any number of quantities, a, b, c, &c., are in *continued* proportion, when

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\&c.$$

x varies *directly* as y when $x=my$;

x varies *inversely* as y when $x=\frac{m}{y}$;

where m is a constant.

The symbol $\propto$ expresses variation. This must be carefully distinguished from ∞, the symbol for an indefinitely large quantity, or infinity.

PROGRESSIONS.

To find the common difference in an A.P.

Subtract any term from the one following.

$$l=a+(n-1)\,d.$$

$$s=\frac{n}{2}\,(l+a)$$

$$=\frac{n}{2}\,\{2a+(n-1)\,d\}.$$

To insert a given number of arithmetical means between two quantities.

Find the common difference by the formula,

$$l=a+(n-1)\,d,$$

where a is one of the given quantities, l the other, and n exceeds the number of means by 2. Then the required means are $a+d$, $a+2d$, $a+3d$, &c.

$$1^2+2^2+3^2+\ldots+n^2=\frac{n\,(n+1)\,(2n+1)}{6}.$$

$$1\times2+2\times3+3\times4+4\times5+\ldots=1^2+2^2+3^2+4^2+\ldots$$
$$+1+2+3+4+\ldots.$$

To find the common ratio in a G.P.

Divide any term by the preceding one.

$$l = ar^{n-1}.$$

$$s_n = \frac{rl - a}{r - 1}$$

$$= \frac{a(r^n - 1)}{r - 1}.$$

$$s_\infty = \frac{a}{1 - r}.$$

To insert a given number of geometrical means between two quantities.

Find the common ratio by the formula,

$$l = ar^{n-1},$$

where a is one of the given quantities, l the other, and n exceeds the number of means by 2. Then the required means will be ar, ar^2, ar^3, &c.

Three quantities, a, b, c, are in H.P., when

$$a : c :: a - b : b - c.$$

The reciprocals of quantities in H.P. are in A.P.

To insert a given number of harmonical means between two quantities.

Insert the same number of arithmetical means between the reciprocals of the given quantities. The reciprocals of these arithmetical means will be the required harmonical means.

If A be the arithmetical mean
$\quad G$ „ geometrical „   between a and b,
$\quad H$ „ harmonical „

$$A = \frac{a+b}{2};$$

$$G = \sqrt{(ab)};$$

$$H = \frac{2ab}{a+b}.$$

When the n^{th} term of a series is given, the different terms beginning with the first are found by making n successively $= 1, 2, 3,$ &c.

SCALES OF NOTATION.

When the	radix	is 2,	the scale	is called	the binary.
"	"	3,	"	"	ternary.
"	"	4,	"	"	quaternary.
"	"	5,	"	"	quinary.
"	"	6,	"	"	senary.
"	"	7,	"	"	septenary.
"	"	8,	"	"	octenary.
"	"	9,	"	"	nonary.
"	"	10,	"	"	denary, or common.
"	"	11,	"	"	undenary.
"	"	12,	"	"	duodenary.

If $d_n, d_{n-1}, \ldots d_2, d_1$ be the digits of a number of n digits in the scale, whose radix is r,

$$\text{the number} = d_n r^{n-1} + d_{n-1} r^{n-2} + \ldots + d_2 r + d_1.$$

The operations of Addition, Subtraction, Multiplication, Division, Square Root, &c., in any scale whose radix is r differ from those in the common scale in the division by r instead of by 10, when the number to be *carried* at any step is to be determined.

To transform a given whole number from one scale to another.

Divide the given number, in the same scale in which it is expressed, by the radix r of the new scale. Again divide the quotient by r, and continue the series of divisions until a quotient less than r is obtained. This last quotient is the extreme left-hand digit of the number in the proposed scale; the last remainder is the next digit, and so on, the first remainder being the extreme right-hand digit.

If $t_1, t_2, t_3, \ldots$ be the digits of a radix-fraction in the scale, whose radix is r,

$$\text{the fraction} = \frac{t_1}{r} + \frac{t_2}{r^2} + \frac{t_3}{r^3} + \ldots.$$

To transform a given radix-fraction from one scale to another.

Multiply the given fraction, in the same scale in which it is expressed, by the radix r of the new scale. Again, multiply the *fractional* part of the result by r, and continue the series of multiplications until a result is obtained, which has no fractional part, or until the fractional parts recur. The integral parts of the successive products will be the digits of the proposed fraction.

PERMUTATIONS AND COMBINATIONS.

The different orders, in which any things can be arranged, are called their *permutations.*

The *combinations* of things are the different collections which can be formed out of them, without regard to the order in which the things stand in each collection.

I.

If there be n processes, the first of which can be performed in a_1 ways, and when any one of these has

been performed, the second can be performed in a_2 ways, and then the third in a_3 ways, and so on, the last being performable in a_n ways, the whole number of ways of performing all the compound processes is

$$a_1 a_2 a_3 \ldots a_n.$$

II.

The number of permutations of n different things taken r at a time

$$= n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{\lfloor n}{\lfloor n-r}.$$

III.

The number of permutations of n different things taken r at a time, when each may be repeated 1, 2, 3, ... up to r times,

$$= n^r.$$

IV.

The number of ways, in which $a_1 + a_2 + a_3 + \ldots + a_n$ things can be divided into n classes containing respectively a_1, a_2, ... a_n things,

$$= \frac{\lfloor a_1 + a_2 + \ldots + a_n}{\lfloor a_1 \lfloor a_2 \ldots \lfloor a_n}.$$

V.

The number of permutations of n things, whereof p are all alike (of one kind), q all alike (of another kind), r all alike (of another kind), and the rest all different,

$$= \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

VI.

The number of combinations of n different things, taken r at a time,

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

VII.

The *total* number of combinations of n different things

$$= 2^n - 1.$$

VIII.

The *total* number of combinations of $p+q+r+\ldots$ things, whereof p are all alike (of one kind), q all alike (of another kind), r all alike (of another kind), &c.,

$$= (p+1)(q+1)(r+1)\ldots-1.$$

The number of combinations of n things taking r together is greatest when,

(i) n being even, $r = \frac{n}{2}$;

(ii) n being odd, $r = \frac{n-1}{2}$, or $\frac{n+1}{2}$.

Def. If an event may happen in a ways, and fail in b ways, and all these ways are equally likely to occur,

the probability of its happening is $\frac{a}{a+b}$,

and " failing is $\frac{b}{a+b}$;

also it is said that the odds are

a to b in favour of the event,

or b to a against the event.

In throwing with two ordinary dice, the total number of throws $= 36$.

7	can be thrown in	6	ways.
6 or 8	"	5	"
5 or 9	"	4	"
4 or 10	"	3	"
3 or 11	"	2	"
2 or 12	"	1	way.

If there be any number of *independent* events, the probability that they will all happen is the product of their respective probabilities of happening.

The probability of the concurrence of two *dependent* events is the product of the probability of the first into the probability that when that has happened the second will follow.

If a person is to receive a prize on condition of some event happening, the sum of money for which his chance might equitably be sold beforehand is called his *expectation* from the event.

The expectation from any event is the product of the sum to be realized on the event happening, and the chance that the event will happen.

BINOMIAL THEOREM.

$$(x+a)^n = x^n + nx^{n-1}a + \frac{n(n-1)}{\underline{|2}} x^{n-2}a^2 + \ldots$$

$$\ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} x^{n-r}a^r + \ldots + nxa^{n-1} + a^n.$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}} x^2 + \ldots$$

$$\ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}} x^r + \ldots + nx^{n-1} + x^n.$$

The general term written in each of the above expansions is the $(r+1)^{\text{th}}$ term, from which we may find any term as the p^{th} by making $r=p-1$.

To find the term, which is the middle term of the expansion of any binomial to the n^{th} power.

If n be even, the $\frac{n+2}{2}$th term is the middle term.

If n be odd, the $\frac{n+1}{2}$th and $\frac{n+3}{2}$th terms are the middle terms.

$(x+a)^n$ can always be reduced to the form $(1+x)^n$;

$$\text{thus } (x+a)^n = a^n\left(1+\frac{x}{a}\right)^n.$$

To find the numerically greatest term in the expansion of $(1+x)^n$.

I.

If n be a positive integer,
the $(q+1)^{\text{th}}$ term is the greatest,
where q is the integral part of $\frac{(n+1)x}{x+1}$.

II.

If n be positive, but not integral,
there is no greatest term if $x>1$;
but if x is not >1,
the $(q+1)^{\text{th}}$ term is the greatest,
where q is the integral part of $\frac{(n+1)x}{x+1}$.

III.

If n be negative, let $n=-m$, so that we have to find the greatest term of $(1+x)^{-m}$.

a. If $x>1$, there is no greatest term.

β. If $x < 1$, and $m > 1$,

the $(q+1)^{\text{th}}$ term is the greatest,

where q is the integral part of $\dfrac{(m-1)x}{1-x}$.

If $x < 1$, and $m < 1$,

the first term is the greatest.

γ. If $x = 1$, and $m > 1$, there is no greatest term.
If $x = 1$, and $m = 1$, all the terms are equal.
If $x = 1$, and $m < 1$, the first term is the greatest.

To find the roots of numbers, which are not exact powers, approximately, by the Binomial Theorem.

Suppose it required to find the n^{th} root of a.

Take b, the nearest number to a, such that $b^{\frac{1}{n}}$ is known exactly, and expand

$$b^{\frac{1}{n}}\left\{1+\frac{a-b}{b}\right\}^{\frac{1}{n}}.$$

The number of homogeneous products of r dimensions that can be formed out of n letters, $a_1, a_2, a_3, \ldots a_n$, and their powers

$$=\frac{n(n+1)\ldots(n+r-1)}{\underline{|r}}.$$

This is, therefore, also the number of terms in the expansion of the multinomial,

$$(a_1+a_2+a_3+\ldots+a_n)^r,$$

where r is a positive integer.

To find the coefficient of x^r in an expression of the form $\dfrac{(1+ax)^n}{(1+bx)^m}$.

Write in the form, $(1+ax)^n(1+bx)^{-m}$, and expand. Take from the latter expansion all those terms, each of which when multiplied by some term in the first ex-

pansion will involve x^r. The algebraic sum of the coefficients of all such products will be the coefficient required.

THEORY OF LOGARITHMS.

If $x = \log_a n$, then $a^x = n$.

$$\log_a 0 = -\infty, \text{ if } a > 1.$$

$$\log 1 = 0.$$

$$\log_a a = 1.$$

$$\log mn = \log m + \log n.$$

$$\log \frac{m}{n} = \log m - \log n.$$

$$\log (m^r) = r \log m.$$

Having given $\log_a n$, *to find* $\log_b n$.

Multiply $\log_a n$ by $\log_b a$.

$\log_{10} e = \cdot 43429448$, and is called the *modulus* of the common system.

$$\log_a b \times \log_b a = 1.$$

If all the logarithms in a table be multiplied by the same number, they will still be logarithms of the same numbers as before, but to a different base.

Let a be the base of the logarithms given in the table.

Let x be any one of these logarithms, and n the corresponding number.

Then $x = \log_a n$;

therefore $a^x = n$;

therefore $(a^{\frac{1}{m}})^{mx} = n$;

therefore $mx = \log_{a^{\frac{1}{m}}} n$.

Hence, if we multiply any logarithm by m, the product will be the logarithm of the same number to a base, which is the m^{th} root of the original base.

$$a^x = 1 + (\log_e a)\, x + \frac{(\log_e a)^2 x^2}{\underline{|2}} + \frac{(\log_e a)^3 x^3}{\underline{|3}} + \dots .$$

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \dots .$$

$$\log x = x - 1 - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 \dots .$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots .$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right).$$

CONVERGENCY AND DIVERGENCY OF SERIES.

DEF. A series consisting of an indefinite number of terms, which are successively formed according to some fixed law, is *convergent*, when the sum of the first n terms approximates to a finite limit as n increases.

The series is said to be *divergent*, when the sum of the first n terms can be made greater than any finite quantity by taking n large enough.

I. A series, in which the terms are all positive, or all negative, is divergent, if each term is > a certain finite quantity.

II. A series represented by

$$u_1 - u_2 + u_3 - u_4 + \&c.,$$

in which the terms are alternately positive and negative, is convergent, if $\frac{u_{n+1}}{u_n}$ be numerically < 1.

III. A series, represented by

$$u_1 + u_2 + u_3 + u_4 + \&c.$$

is convergent if, for all values of n beyond a certain number m, $\frac{u_{n+1}}{u_n}$ be numerically $<$ a proper fraction k.

If $\frac{u_{n+1}}{u_n} > 1$, the series is divergent.

CONTINUED FRACTIONS.

Continued Fraction is the name given to an expression of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.,}}}$$

where a, b, c, d, &c. are positive integers, the number of which may be finite or not.

The terms a, b, c, &c. are called *quotients*, and when their number is finite, the continued fraction is said to be *terminating*.

If in a continued fraction we take successively one, two, three, ... of the quotients, and reduce the expressions to the form of vulgar fractions, we obtain *converging fractions*, or *convergents*.

The 1st convergent $<$ the continued fraction;

the 2nd convergent $>$ the continued fraction;

and, generally,

any convergent $> <$ the continued fraction,

according as it is of the even or the odd rank.

If $\frac{p}{q}$, $\frac{p'}{q'}$, $\frac{p''}{q''}$ be three consecutive convergents, m, m', m'' the corresponding quotients,

$$p'' = m''p' + p,$$

$$q'' = m''q' + q,$$

$$\frac{p''}{q''} - \frac{p'}{q'} = \pm \frac{1}{q''q'},$$

the upper or lower sign being taken according as the convergent $\frac{p''}{q''}$ is of the even or the odd rank.

To convert a vulgar fraction into a continued fraction.

Let the fraction be $\frac{N}{D}$.

(α) If $N > D$, divide N by D; let the quotient be a, and the remainder r_1, so that

$$\frac{N}{D} = a + \frac{r_1}{D} = a + \frac{1}{\frac{D}{r_1}}.$$

Next, divide D by r_1; let the quotient be b, and the remainder r_2; therefore

$$\frac{N}{D} = a + \frac{1}{b+} \frac{r_2}{r_1} = a + \frac{1}{b+} \frac{1}{\frac{r_1}{r_2}}.$$

Continue this series of operations, *which is the same as that for finding the* G.C.M. *of N and D*, until the remainder is zero.

(β) If $N < D$, write the fraction in the form

$$\frac{1}{\frac{D}{N}};$$

divide D by N, and let a be the quotient, and r_1 the remainder; thus

$$\frac{N}{D} = \frac{1}{a+} \frac{r_1}{N} = \frac{1}{a+} \frac{1}{\frac{N}{r_1}}, \text{ and so on.}$$

To reduce a quadratic surd to a continued fraction.

Let the given surd be $\sqrt{(N)}$.

Let a be the greatest integer in $\sqrt{(N)}$.

Then $$\sqrt{(N)} = a + \frac{\sqrt{(N)} - a}{1}.$$

Rationalise the numerator; thus

$$\sqrt{(N)} = a + \frac{N - a^2}{\sqrt{(N)} + a}$$

$$= a + \frac{1}{\dfrac{\sqrt{(N)} + a}{N - a^2}}.$$

Now let b be the greatest integer in $\dfrac{\sqrt{(N)} + a}{N - a^2}$; thus

$$\sqrt{(N)} = a + \cfrac{1}{b + \dfrac{\sqrt{(N)} + a - b\,(N - a^2)}{N - a^2}}.$$

Rationalise the numerator, and continue the series of operations as far as required.

INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

The general form of indeterminate equations of the first degree is

$$ax \pm by = c,$$

where a, b, and c are integers, which have no common factor, such factors having been got rid of by division.

Excluding *zero* values of x and y,

$$ax + by = c$$

cannot have any integral solution if

$$a + b > c.$$

Given one solution in positive integers,

$$x = \alpha,$$

$$y = \beta,$$

of the equation

$$ax + by = c,$$

all the solutions can be obtained by giving to t different integral values in the equations

$$x = \alpha + bt,$$
$$y = \beta - at.$$

Given one solution in positive integers,

$$x = \alpha,$$
$$y = \beta,$$

of the equation

$$ax - by = c,$$

all the solutions can be found by giving to t different integral values in the equations

$$x = \alpha + bt,$$
$$y = \beta + at.$$

To solve an indeterminate equation

$$ax \pm by = c.$$

Divide both sides of the equation by the smaller of the two coefficients, a, b, and convert the improper fractions in the result into mixed numbers.

Transpose the equation, placing the integral parts on one side of the equation, and the fractional parts on the other side.

Put the fractional parts equal to an integer p, and clear this equation of fractions.

Treat the result in the same way as the original equation, introducing a new integer, q.

Repeat the operation, introducing a new integer at each step, until the coefficient of one of the unknown quantities is 1, and that of the other an integer.

Substitute in the preceding equations, and thus obtain expressions for the original unknown quantities in terms of the last letter introduced.

To solve the indeterminate equation,

$$ax + by + cz = d.$$

Transfer the term, which has the largest coefficient, to the right-hand side of the equation.

Let this term be cz; thus

$$ax + by = d - cz.$$

Assign to z different integral values, such that

$$d - cz \text{ is positive,}$$

$$\text{and not } < a + b.$$

Solve the various resulting indeterminate equations.

To solve the indeterminate equation,

$$ax + by = c,$$

by the use of continued fractions.

Reduce $\frac{a}{b}$ to a continued fraction, and calculate all the convergents, the last of which is $\frac{a}{b}$ itself.

Let $\frac{a'}{b'}$ be the last convergent but one.

Then, if $$ab' - a'b = +1,$$

$$x = b'c - bt, \quad y = at - a'c.$$

If $$ab' - a'b = -1,$$

$$x = bt - b'c, \quad y = a'c - at.$$

INDETERMINATE COEFFICIENTS.

If the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = 0,$$

whatever be the value of x, then the coefficients a_0, a_1, a_2, ... must each separately $= 0$.

The following are some of the applications of the principle of indeterminate coefficients:

I. The expansion of rational fractions.

II. The decomposition of rational fractions into their partial fractions.

III. The summation of series.

I.

To expand the fraction $\dfrac{a_0 + a_1x}{b_0 + b_1x + b_2x^2}$ *in ascending powers of* x.

This may be done in three ways.

(i) By ordinary algebraic division.

(ii) By the expansion of $(b_0 + b_1x + b_2x^2)^{-1}$ by the Binomial Theorem.

(iii) By indeterminate coefficients as follows.

Suppose $\dfrac{a_0 + a_1x}{b_0 + b_1x + b_2x^2} = u_0 + u_1x + u_2x^2 + u_3x^3 + \dots$.

Multiply both sides by the denominator of the fraction; thus

$$a_0 + a_1x = (b_0 + b_1x + b_2x^2)(u_0 + u_1x + u_2x^2 + \dots).$$

Equate the coefficients of the powers of x on one side of the equation with those of the corresponding powers of x on the other side, thus obtaining equations for determining u_0, u_1, &c.

II.

To decompose a given rational fraction into its partial fractions.

If the numerator be not of a lower degree than the denominator, reduce the fraction to a mixed quantity by division.

(α) Let the denominator be the product of n different factors of the first degree; thus the fraction may be represented by

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)},$$

where $f(x)$ is of a lower degree than the denominator.

Assume

$$\frac{f(x)}{(x-a_1)(x-a_2)\ldots(x-a_n)}=\frac{N_1}{x-a_1}+\frac{N_2}{x-a_2}+\ldots+\frac{N_n}{x-a_n}.$$

Multiply by $x-a_1$, and put $x=a_1$ in the resulting equation; thus N_1 is found.

Again, multiply by $x-a_2$, and put $x=a_2$ in the result; thus N_2 is found.

Similarly, N_3, N_4, &c. can be found.

(β) Let any number m of the factors of the denominator be the same; then the fraction will be of the form

$$\frac{f(x)}{(x-a_1)^m Q},$$

where Q represents the product of the unequal factors, $x-a_2$, $x-a_3$, &c.

Assume

$$\frac{f(x)}{(x-a_1)^m Q}=\frac{M_1}{x-a_1}+\frac{M_2}{(x-a_1)^2}+\ldots$$
$$+\frac{M_m}{(x-a_1)^m}+\frac{N_2}{x-a_2}+\frac{N_3}{x-a_3}+\ldots.$$

Multiply by $(x-a_1)^m$; put $x=a_1$ in the result, and thus determine M_m.

Again, multiply by $x-a_2$; put $x=a_2$ in the result, and thus determine N_2.

Similarly N_3, N_4, &c. can be found.

To find the remaining numerators M_1, M_2, ... M_{m-1}, multiply both sides by $(x-a_1)^m Q$, substitute for M_m,

N_2, N_3, ... their values, and equate coefficients of like powers of x in the resulting equation.

(γ) Let the denominator contain a quadratic factor, x^2-px+q, which can only be resolved into imaginary factors.

Assume
$$\frac{f(x)}{(x^2-px+q)\,Q}=\frac{Rx+S}{x^2-px+q}+\frac{P}{Q},$$
where $\frac{P}{Q}$ represents the sum of the partial fractions, which can be determined by (α) and (β). Having determined these, multiply by $(x^2-px+q)\,Q$; equate coefficients, and find R and S from the resulting equations.

III.

To find S, where

$$S=a^m+(a+d)^m+(a+2d)^m+\text{\&c. to } n \text{ terms.}$$

The n^{th} term is $\{a+(n-1)\,d\}^m$.

Assume
$$a^m+(a+d)^m+\ldots\{a+(n-1)\,d\}^m=A_0n^{m+1}+A_1n^m+\ldots+A_mn \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$
Then
$$a^m+(a+d)^m+\ldots+(a+nd)^m$$
$$=A_0(n+1)^{m+1}+A_1(n+1)^m+\ldots+A_m(n+1)\;\ldots\ldots.(2).$$

Subtract (1) from (2), and equate coefficients of corresponding powers of n.

To find S, where

$$S=a\,(a+d)+(a+d)\,(a+2d)+\ldots \textit{ to } n \textit{ terms.}$$

The n^{th} term is $\{a + (n-1)\,d\}\,\{a + nd\}$.

Assume $\quad S = A_0 n^3 + A_1 n^2 + A_2 n$.

Let $\quad S_1 =$ the sum of $(n+1)$ terms; then

$$S_1 = A_0\,(n+1)^3 + A_1\,(n+1)^2 + A_2\,(n+1).$$

Therefore $S_1 - S = (a + nd)\,\{a + (n+1)\,d\}$

$$= A_0\,\{(n+1)^3 - n^3\} + A_1\,\{(n+1)^2 - n^2\} + A_2.$$

Equate coefficients of like powers of n.

To find S, where

$$S = 1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + n\,(n+1)\,(n+2).$$

Assume $S = A_0 n^4 + A_1 n^3 + A_2 n^2 + A_3 n$

Let $\quad S_1 =$ the sum of $(n+1)$ terms; then

$$S_1 = A_0 (n+1)^4 + A_1 (n+1)^3 + A_2 (n+1)^2 + A_3 (n+1).$$

Therefore

$$S_1 - S = (n+1)\,(n+2)\,(n+3)$$

$$= A_0\{(n+1)^4 - n^4\} + A_1\{(n+1)^3 - n^3\} + A_2\{(n+1)^2 - n^2\} + A_3.$$

Equating coefficients,

$1 = 4A_0$; $\quad$ therefore $A_0 = \frac{1}{4}$.

$6 = 6A_0 + 3A_1$; $\quad A_1 = \frac{3}{2}$.

$11 = 4A_0 + 3A_1 + 2A_2$; $\quad A_2 = \frac{11}{4}$.

$6 = A_0 + A_1 + A_2 + A_3$; $\quad A_3 = \frac{3}{2}$.

Thus $\quad S = \frac{1}{4}\,(n^4 + 6n^3 + 11n^2 + 6n)$

$$= \frac{n}{4}\,(n+1)\,(n+2)\,(n+3).$$

This and similar series may be more readily summed by the *method of subtraction.*

Let $S = 1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + n\,(n+1)\,(n+2)$.

Take one factor more in each term, and let

$$S_1 = 1 \times 2 \times 3 \times 4 + 2 \times 3 \times 4 \times 5 + \ldots + n(n+1)(n+2)(n+3);$$

$$S_1 = \qquad 1 \times 2 \times 3 \times 4 + \ldots + (n-1)n(n+1)(n+2)$$
$$+ n(n+1)(n+2)(n+3);$$

subtracting,

$$0=4\{1\times2\times3+2\times3\times4+\ldots+n(n+1)(n+2)\}$$
$$-n(n+1)(n+2)(n+3);$$

that is, $\quad 0=4S-n(n+1)(n+2)(n+3);$

therefore $\quad S=\frac{n(n+1)(n+2)(n+3)}{4}.$

Let $S=\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\ldots+\frac{1}{n(n+1)(n+2)}.$

Take one factor less in each term, and let

$$S_1=\frac{1}{1\times2}+\frac{1}{2\times3}+\ldots+\frac{1}{(n+1)(n+2)};$$

$$S_1=\qquad\frac{1}{1\times2}+\ldots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)};$$

subtracting,

$$0=\frac{1}{1\times2}-2\left\{\frac{1}{1\times2\times3}+\ldots+\frac{1}{n(n+1)(n+2)}\right\}-\frac{1}{(n+1)(n+2)};$$

that is, $\quad 0=\frac{1}{1\times2}-2S-\frac{1}{(n+1)(n+2)};$

therefore $\quad 2S=\frac{1}{2}-\frac{1}{(n+1)(n+2)};$

therefore $\quad S=\frac{n(n+3)}{4(n+1)(n+2)}.$

DIVISION.

Division is the process by which, when a product and one of its factors are given, the other factor is determined.

The factor to be determined is called the Quotient.

The given factor is called the Divisor.

The product is called the Dividend.

The process of division is the first which presents any serious difficulty to the beginner in Algebra, but

this seems to arise mainly from the neglect of the following important rule.

The dividend and divisor must be arranged according to descending or ascending powers of some letter common to both; and each remainder must be arranged in the same order.

Now in the text books at present in use the examples are invariably arranged in this order, so that the pupil's knowledge of this rule is rarely tested; and it is by no means an uncommon thing in school examinations to see two or three sheets of paper filled with futile attempts to work out a division by one ignorant or careless of this arrangement. Especially is this the case, when the divisor does not exactly measure the dividend, as the unfortunate, having always had divisions which "came out," labours on hopefully, yet vainly, to finish by having no remainder.

We shall then confine our attention to divisions in which the quotient is interminable.

I. The ordinary process of division shews that the first and last terms of the dividend when properly arranged and those of each remainder ought to be respectively divisible by the first and last terms of the divisor, in order that the division may terminate.

For example, suppose we have to divide

$$14x^4 + 15x^5 - 15x^3 + 3x + 8x^2 + 6 \text{ by } 1 - 2x + 5x^2.$$

Arrange both divisor and dividend according to *decreasing* powers of x, and divide; thus

$$\begin{array}{rl}
5x^2-2x+1) & 15x^5 + 14x^4 - 15x^3 + 8x^2 + 3x + 6 \; (3x^3+4x^2-2x \\
 & \underline{15x^5 - 6x^4 + 3x^3} \\
 & \phantom{15x^5 + {}} 20x^4 - 18x^3 + 8x^2 \\
 & \phantom{15x^5 + {}} \underline{20x^4 - 8x^3 + 4x^2} \\
 & - 10x^3 + 4x^2 + 3x \\
 & \underline{- 10x^3 + 4x^2 + 2x} \\
 & 5x + 6
\end{array}$$

When we come to the remainder $5x+6$ the division stops, for the first term $5x$ is not divisible by $5x^2$, the first term of the divisor. Hence it is impossible to find the quotient in an *algebraically integral* form without any remainder.

II. Again arrange both divisor and dividend according to *increasing* powers of x, and divide:

$$\begin{array}{r|l}
1-2x+5x^2) & 6+3x+8x^2-15x^3+14x^4+15x^5\ (6+15x+8x^2-74x^3 \\
 & \underline{6-12x+30x^2} \\
 & \quad 15x-22x^2-15x^3 \\
 & \quad \underline{15x-30x^2+75x^3} \\
 & \qquad 8x^2-90x^3+14x^4 \\
 & \qquad \underline{8x^2-16x^3+40x^4} \\
 & \qquad\quad -74x^3-26x^4+15x^5 \\
 & \qquad\quad \underline{-74x^3+148x^4-370x^5} \\
 & \qquad\qquad -174x^4+385x^5
\end{array}$$

By continuing this process we obtain remainders of which the first and last terms are always divisible by the first and last terms of the divisor, so that in this case the foregoing test for the termination of the division does not hold good.

But we know that if the division be terminating, the last term of the dividend is the exact product of the last term of the divisor and the last term of the quotient; that is to say, we always know beforehand what the last term of the quotient ought to be. In our example it ought to be $\frac{15x^5}{5x^2}=3x^3$.

If, then, in the process of division we obtain in the quotient a term similar to this predetermined last term, but differing from it in value or in sign; or again, if, in case of this term being equal both in value and sign to the anticipated term, the next remainder is not zero, we may be sure that it never will be, and that if we continue the division, we shall obtain as quotient a polynomial consisting of as many terms as we please.

In the above division, instead of $3x^3$ which ought in case of the quotient being terminable to have been the last term, we obtained $-74x^3$, so that the division will never terminate.

To sum up, then, we have the two following tests:

I. If we arrange according to *descending* powers, the degree of the first term of each remainder is one less than that of the first term of the preceding remainder, and if the division does not terminate when the dimensions of the remainder become equal to those of the divisor, the division will never terminate.

II. If we arrange according to *ascending* powers, we must determine beforehand what ought to be the last term of the quotient; then if we obtain in the quotient a term similar to this, but with a different coefficient, or if on the other hand we obtain a term equal to it and the following remainder is not zero, we know that the division will never terminate.

GREATEST COMMON MEASURE.

It does not necessarily follow that the Highest Common Factor of two algebraical quantities always gives the Greatest Common Measure of the numbers, which result when particular numbers are substituted for the algebraical symbols.

Let the given expressions be denoted by a and b, and let their H.C.F. be h.

Then h is the expression of highest dimensions with respect to some letter common to a and b, which will exactly divide a and b.

Let α and β be the products of the remaining factors in a and b respectively, so that

$$a = \alpha h,$$

$$b = \beta h.$$

Then α and β have no *algebraical* common factor; yet, when particular numbers are substituted for the symbols, they may have a *numerical* common factor, and in this case the arithmetical G.C.M. will be the product of the numerical value of h and this common factor.

RATIO.

A ratio of greater inequality is diminished, and a ratio of less inequality is increased by adding the same quantity to both terms of the ratio.

Let the ratio be represented by the fraction $\frac{a}{b}$,

and let x be added to both terms of the ratio;

then $$\frac{a+x}{b+x} > < \frac{a}{b},$$

according as $$\frac{b(a+x)}{b(b+x)} > < \frac{a(b+x)}{b(b+x)},$$

as $$b(a+x) > < a(b+x),$$

as $$ab + bx > < ab + ax,$$

as $$bx > < ax,$$

as $$b > < a.$$

A ratio of greater inequality is increased, and a ratio of less inequality is diminished by taking from both terms of the ratio the same quantity which is less than each of those terms.

Let the ratio be represented by the fraction $\frac{a}{b}$,

and let x, a quantity less than either a or b, be taken from both terms of the ratio;

then $$\frac{a-x}{b-x} > < \frac{a}{b},$$

according as $$\frac{b(a-x)}{b(b-x)} > < \frac{a(b-x)}{b(b-x)},$$

as $$b(a-x) >< a(b-x),$$

as $$ab - bx >< ab - ax,$$

as $$-bx >< -ax,$$

as $$-b >< -a,$$

as $$b <> a.$$

QUADRATIC EQUATIONS.

The most general equation of the second degree with one unknown (x) consists of three sorts of terms;

(i) terms which involve x^2;

(ii) terms which involve x;

(iii) terms entirely composed of known quantities.

If we transpose all the terms to one side, combining those which involve x^2 into one term, those which involve x into another term, and all the others into a third term, we may represent the equation thus:

$$ax^2 + bx + c = 0,$$

where a stands for the collected coefficient of x^2, b for that of x, and c for all the terms entirely composed of known quantities.

We briefly express the above by saying that the most general form of a quadratic is

$$ax^2 + bx + c = 0 \dots\dots\dots\dots\dots\dots(1).$$

To solve (1) we transpose c to the other side;

thus $$ax^2 + bx = -c;$$

dividing by a, $$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

If necessary, we now change all the signs, so that the coefficient of x^2 may be $+1$.

Then completing the square by adding to each side the square of half the coefficient of x, we have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2};$$

extracting the square root,

$$x + \frac{b}{2a} = \pm \frac{\sqrt{(b^2 - 4ac)}}{2a};$$

therefore, $$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Thus, if we represent the two roots by α and β,

$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}, \quad \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a} \quad \text{........(2)};$$

multiply numerator and denominator in the value of α by $-b - \sqrt{(b^2 - 4ac)}$; thus

$$\alpha = \frac{\{-b + \sqrt{(b^2 - 4ac)}\}\{-b - \sqrt{(b^2 - 4ac)}\}}{2a\{-b - \sqrt{(b^2 - 4ac)}\}}$$

$$= \frac{b^2 - (b^2 - 4ac)}{2a\{-b - \sqrt{(b^2 - 4ac)}\}}$$

$$\left.\begin{aligned} &= \frac{2c}{-b - \sqrt{(b^2 - 4ac)}}. \\ \text{Similarly,} \quad \beta &= \frac{2c}{-b + \sqrt{(b^2 - 4ac)}}. \end{aligned}\right\} \text{.................(3).}$$

From (2) we have

$$\alpha + \beta = \frac{-b + \sqrt{(b^2 - 4ac)} - b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$= -\frac{b}{a};$$

$$\alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

To shew that if α and β be the roots of the equation,

$$ax^2 + bx + c = 0,$$

then $$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

Since $$\alpha + \beta = -\frac{b}{a},$$

$$b = -a(\alpha + \beta);$$

and $$\alpha\beta = \frac{c}{a};$$

therefore $$c = a(\alpha\beta);$$

therefore substituting,

$$\begin{aligned} ax^2 + bx + c &= ax^2 - a(\alpha + \beta)x + a(\alpha\beta) \\ &= a\{x^2 - (\alpha + \beta)x + \alpha\beta\} \\ &= a(x - \alpha)(x - \beta). \end{aligned}$$

Let us now consider the values of the roots of (1) under certain circumstances.

(i) If $$b^2 > 4ac,$$

$b^2 - 4ac$ is a positive quantity;

therefore, $\sqrt{(b^2 - 4ac)}$ is a real or possible quantity;

therefore, from (2), the two roots are real and different.

(ii) If $$b^2 = 4ac,$$

$$b^2 - 4ac = 0;$$

therefore, from (2), the two roots are real and equal.

(iii) If $$b^2 < 4ac,$$

$b^2 - 4ac$ is a negative quantity;

therefore, $\sqrt{(b^2 - 4ac)}$ is an imaginary or impossible quantity;

therefore, from (2), both the roots are imaginary.

(iv) If $$a = 0,$$
we see by substituting in (3) that

one root is ∞, and the other $-\dfrac{c}{b}$.

(v) If $$a = 0, \text{ and } b = 0,$$
each root $= \infty$, by (3).

(vi) If $$b \text{ only} = 0,$$
since $\alpha + \beta = -\dfrac{b}{a}$;

therefore, $\alpha = -\beta$;

that is, the two roots are equal, but of opposite sign.

(vii) If $$b = 0, \text{ and } c = 0,$$
each root $= 0$, by (2).

(viii) If $$c \text{ only} = 0,$$
we see from (2) that

one root $= 0$, and the other $-\dfrac{b}{a}$.

GEOMETRICAL PROGRESSION.

DEF. The limit of a variable quantity is that quantity to which it may be made continually to approximate, but to which it never becomes actually equal, although the difference may be made less than any assignable.

The variable quantity is said to be ultimately equal to the limit.

To illustrate this,

let $S =$ the sum of n terms of the geometrical series,

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \dots;$$

then $$S = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

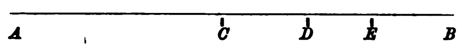
Now, by continually increasing n, we make S continually approximate to 2;

and the difference between S and 2, that is $\frac{1}{2^{n-1}}$, can be made less than any assignable quantity by taking n large enough;

therefore, S is ultimately equal to 2, when n is indefinitely great.

Let us further illustrate this geometrically.

Take a straight line AB to represent 2 units.



Bisect it in C; again bisect CB in D, DB in E, and so on.

Now AC, CD, DE, ... represent $1, \frac{1}{2}, \frac{1}{4}, \ldots$, respectively,

and their sum is seen to approximate to AB,

but whatever degree of approximation may be made to the point B by passing from A to C, from C to D, from D to E, &c., it is clear that as much remains to be passed over as was passed over at the last step, though we can make this distance to be passed over less than any finite distance that can be assigned by continuing the process of bisection.

NOTATION OF FUNCTIONS.

One quantity is said to be a function of another when any change in the latter produces a corresponding variation in the former.

Thus $ax^2 + bx$, $\sin x$, a^x are all functions of x, and may each be denoted for shortness by the symbol $f(x)$, $\phi(x)$, or $F(x)$.

An algebraical function of x is an expression which

consists of a finite number of terms, involving powers of x together with constants.

Thus ax^2+bx+c is an algebraical function of x, and ay^2+by+c is the same function of y, so that if we denote ax^2+bx+c by $f(x)$, ay^2+by+c will be denoted by $f(y)$.

A function of x is said to be rational and integral when it does not contain x in the denominator of any term, and when the indices of the powers of x are positive whole numbers.

Example.

If $$f(a, x) = x^2 + ax + a^2,$$

shew that $$f(a, x) . f(a, -x) = f(a^2, x^2).$$

We have $f(a, x) = x^2 + ax + a^2$;

therefore $f(a, -x) = (-x)^2 + a(-x) + a^2 = x^2 - ax + a^2$;

therefore

$$\begin{aligned} f(a, x) . f(a, -x) &= (x^2+ax+a^2)(x^2-ax+a^2) \\ &= x^4 + a^2x^2 + a^4 \\ &= f(a^2, x^2). \end{aligned}$$

Examples.

1. If $$f(x) = 5x - 8,$$

write down the values of $f(0)$, $f(1\frac{3}{5})$, $f(2)$.

2. If $$f(x) = 5x^3 + 6x^2,$$

find $$f(1),\ f(-2).$$

3. If $$f(x) = x + 6,$$

what is the value of $f(x-3)$?

4. If $$f(x) = 6x - 8,$$

find $$y, \text{ when } f(y) = 0.$$

5. If $$f(x) = 2x^2 - 7x + 6,$$
find the values of x, which make $f(x) = 0$.

6. If $$F(x) = ax + b,$$
what is the value of $F(x) . F(y)$?

7. Shew that $f(x) . f(-x) = f(-x^2)$,
when $$f(x) = 1 + x.$$

8. If $$f(x) = x + \frac{1}{x},$$
shew that $$f(x^2) + 2 = \{f(x)\}^2.$$

9. If $$f(x, y) = x + y,$$
find x and y, when
$$f(x, y) = 96, \ f(x, -y) = 2.$$

10. Find x and y, having given
$$\phi(x, y) = x + y = 12;$$
$$\phi(x^2, y^2) = 104.$$

11. Find x and y, having given
$$f(x, y) = x - y;$$
$$f\left(\frac{1}{x}, \frac{1}{y}\right) = 3;$$
$$f\left(\frac{1}{x^2}, \frac{1}{y^2}\right) = 21.$$

12. If $$f(x) = (1 - 2x - x^2)^2,$$
shew that $f(x) - f(-x) = 8x(x-1)(x+1)$.

13. If $$f(a, b) = a^2 + 2ab + b^2,$$
what is the value of $f(-a, -b)$?
Shew that $$f(-a, b) = f(a, -b).$$

14. Having given $f(x) = \frac{1}{x}$,

and $f(x) = f(y) = f(z)$,

Shew that $f(x) = f(x - y + z)$

$$= 3f(x + y + z).$$

15. If $f(x) = \frac{1}{x}$,

and $f(x) = f(y) = f(z)$,

shew that $f(x - y + z) = f(y - z + x) = f(z - x + y)$

$$= -f(x - y - z).$$

16. If $f(x) = (1 + x)^n$,

write down the first four terms of

$$f(x) - f(-x).$$

17. If $f(n) = (1 + a)^n$,

shew that $f(n).f(m) = f(n + m)$;

and that $\{f(n)\}^{\frac{1}{m}} = f\left(\frac{n}{m}\right)$.

18. Having given $f(y) = \log y$,

shew that $f(y) + f(z) = f(yz)$,

$$f(y) - f(z) = f\left(\frac{y}{z}\right).$$

19. If

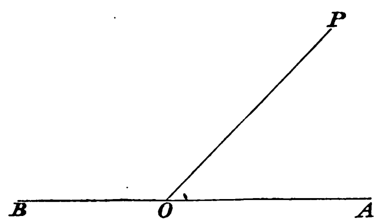
$$F(n, m) = \frac{1}{m} - n.\frac{1}{m+p} + \frac{n(n-1)}{1.2}.\frac{1}{m+2p} - \text{\&c.},$$

shew that

$$F(n, m) = \frac{np}{m}.F\{(n-1), (m+p)\}.$$

PLANE TRIGONOMETRY.

Def. Let BOA be a fixed straight line, and let the line OP, which at first coincides with OA, revolve about the fixed point O.



When the revolving line has reached the position OP, it is said to have described *the angle* POA.

Def. Unity is that magnitude, which we reckon as one, when other magnitudes of the same kind are to be measured by comparison with it.

There are three units of angular measurement:

I. The degree, which is the ninetieth part of a right angle.

II. The grade, which is the one-hundredth part of a right angle.

III. The unit of circular measure, which is the angle subtended at the centre of a circle by an arc equal in length to the radius.

Def. The circular measure of an angle is the number expressing the ratio of the angle to the angle subtended at the centre of a circle by an arc equal in length to the radius.

If at the centre of a circle of radius r an angle be subtended by an arc, whose length is l, its circular measure $= \dfrac{l}{r}$.

Thus the circular measure of four right angles is $\dfrac{2\pi r}{r}$, that is, 2π.

To reduce degrees, minutes, and seconds to grades, minutes, and seconds, or vice versâ.

Express in decimals of a degree or grade, and use the formula,

$$D = \tfrac{9}{10} G, \quad \text{or} \quad G = \tfrac{10}{9} D.$$

If an angle, whose circular measure is θ, contain x° and y^g,

$$\frac{x}{180} = \frac{y}{200} = \frac{\theta}{\pi}.$$

If x be the number of degrees in an angle of a regular polygon of n sides,

$$x = \frac{180(n-2)}{n};$$

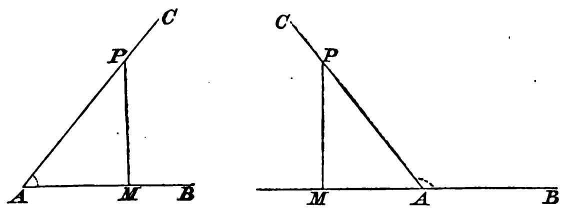
$$n = \frac{360}{180 - x}.$$

DEF. If two angles are together equal to a right angle, each of them is called the *complement* of the other.

DEF. If two angles are together equal to two right angles, each of them is called the *supplement* of the other.

The principal trigonometrical functions or ratios of an angle are the sine, cosine, tangent, cotangent, secant, cosecant, versed sine, and coversed sine. They are defined as follows:

Let BAC be any angle A; in either AB or AC take any point P, and draw PM perpendicular to the other, produced if necessary.



Then $\sin A = \frac{PM}{AP}$. $\operatorname{cosec} A = \frac{AP}{PM}$.

$\cos A = \frac{AM}{AP}$. $\sec A = \frac{AP}{AM}$.

$\tan A = \frac{PM}{AM}$. $\cot A = \frac{AM}{PM}$.

$\operatorname{vers} A = 1 - \cos A$. $\operatorname{covers} A = 1 - \sin A$.

$$\sin A = \frac{1}{\operatorname{cosec} A}.$$

$$\cos A = \frac{1}{\sec A}.$$

$$\tan A = \frac{1}{\cot A} = \frac{\sin A}{\cos A}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\sec^2 A = 1 + \tan^2 A.$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

$$\sin(n360° + A) = \sin A.$$

$$\cos(n360° + A) = \cos A.$$

$$\sin(90° - A) = \cos A.$$

$$\cos(90° - A) = \sin A.$$

$$\sin(90^\circ + A) = \cos A.$$
$$\cos(90^\circ + A) = -\sin A.$$

$$\sin(180^\circ - A) = \sin A.$$
$$\cos(180^\circ - A) = -\cos A.$$

$$\sin(180^\circ + A) = -\sin A.$$
$$\cos(180^\circ + A) = -\cos A.$$

$$\sin(-A) = -\sin A.$$
$$\cos(-A) = \cos A.$$

$\sin 0^\circ = 0.$	$\cos 0^\circ = 1.$
$\sin 30^\circ = \frac{1}{2}.$	$\cos 30^\circ = . \ \frac{\sqrt{3}}{2}$
$\sin 45^\circ = \frac{1}{\sqrt{2}}.$	$\cos 45^\circ = \frac{1}{\sqrt{2}}.$
$\sin 60^\circ = \frac{\sqrt{3}}{2}.$	$\cos 60^\circ = \frac{1}{2}.$
$\sin 90^\circ = 1.$	$\cos 90^\circ = 0.$
$\sin 180^\circ = 0.$	$\cos 180^\circ = -1.$

General expression for all angles which have

(i) a given sine, or cosecant,

$$n\pi + (-1)^n \alpha;$$

(ii) a given cosine, or secant,

$$2n\pi \pm \alpha;$$

(iii) a given tangent, or cotangent,

$$n\pi + \alpha.$$

$$\sin(P+Q) = \sin P \cos Q + \cos P \sin Q.$$
$$\sin(P-Q) = \sin P \cos Q - \cos P \sin Q.$$
$$\cos(P+Q) = \cos P \cos Q - \sin P \sin Q.$$
$$\cos(P-Q) = \cos P \cos Q + \sin P \sin Q.$$

$$2\sin A\cos B=\sin(A+B)+\sin(A-B).$$
$$2\cos A\sin B=\sin(A+B)-\sin(A-B).$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B).$$
$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$
$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$
$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}.$$
$$\cos Q-\cos P=2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}.$$

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$$
$$\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$$
$$=\cos^2 B-\sin^2 A.$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

$$\sin 2A=2\sin A\cos A.$$
$$\cos 2A=\cos^2 A-\sin^2 A$$
$$=1-2\sin^2 A$$
$$=2\cos^2 A-1.$$

$$1-\cos 2A=2\sin^2 A.$$
$$1+\cos 2A=2\cos^2 A.$$

Thus $$\frac{\sin 2A}{1+\cos 2A}=\tan A,$$

and $$\frac{1-\cos 2A}{1+\cos 2A}=\tan^2 A.$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}.$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}.$$

$$\sin 3A = 3\sin A - 4\sin^3 A.$$

$$\cos 3A = 4\cos^3 A - 3\cos A.$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}.$$

$$\cos\tfrac{1}{2}A + \sin\tfrac{1}{2}A = \pm\sqrt{(1+\sin A)}.$$

$$\cos\tfrac{1}{2}A - \sin\tfrac{1}{2}A = \pm\sqrt{(1-\sin A)}.$$

DEF. The symbol $\sin^{-1}x$ denotes *an angle* whose sine is x, and a similar notation is used with the other ratios.

$$Lt._{\theta=0}\left(\frac{\sin\theta}{\theta}\right) = 1.$$

$$Lt._{\theta=0}(\cos\theta) = 1.$$

$$Lt._{m=\infty}\left(m\sin\frac{\theta}{m}\right) = \theta.$$

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}. \qquad \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4}. \qquad \cos 18^\circ = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$$

$$\sin 36^\circ = \frac{\sqrt{(10-2\sqrt{5})}}{4}. \qquad \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$$

If an angle be greater than 90°, its trigonometrical ratios may be reduced to those of an angle less than 90° by the use of the formulæ on pp. 77, 78.

In the first quadrant the logarithms of the
sine, tangent, and secant *increase* as the angle increases; those of the
cosine, cotangent, cosecant *decrease* as the angle increases.

$$L \sin A = \log \sin A + 10.$$

The tabular difference is the difference between the logarithmic functions of two angles, which differ by $1'$ or $60''$.

When A contains degrees and minutes only, to find $L \sin A$, $L \cos A$, &c. from the tables.

When $A < 45°$, look for the degrees at the top of the page and the minutes on the left-hand side.

When $A > 45°$, look for the degrees at the bottom of the page and the minutes on the right-hand side.

The required results are immediately furnished by the tables.

When A contains degrees, minutes, and seconds, to find $L \sin A$, $L \tan A$, or $L \sec A$ from the tables.

Find the logarithm for the number of degrees and minutes; multiply the tabular difference by the number of seconds; divide the product by 60, and *add* the result to the logarithm already found.

When A contains degrees, minutes, and seconds, to find $L \cos A$, $L \cot A$, or $L \operatorname{cosec} A$ from the tables.

Find the logarithm for the number of degrees and minutes; multiply the tabular difference by the number of seconds; divide the product by 60, and *subtract* the result from the logarithm already found.

When the tables cannot be used we have given the logarithmic functions of two angles, between which the given angle lies.

To find $L \sin A$, $L \tan A$, or $L \sec A$, without the use of the tables.

(i) Take the difference between the two given angles.

(ii) Take the difference between their logarithms.

(iii) Subtract the smaller of the two angles from A.

Then multiply the logarithmic difference (ii) by the result of (iii), and divide the product by the difference (i); *add* the result to the logarithm of the smaller of the two given angles.

To find $L \cos A$, $L \cot A$, or $L \operatorname{cosec} A$, without the use of the tables.

(i) Take the difference between the two given angles.

(ii) Take the difference between their logarithms.

(iii) Subtract the smaller of the two angles from A.

Then multiply the logarithmic difference (ii) by the result of (iii), and divide the product by the difference (i); *subtract* the result from the logarithm of the smaller of the two given angles.

To find the angle which corresponds to a given logarithmic sine, tangent, or secant.

Find the next *lower* logarithm in the tables, and take the number of degrees and minutes corresponding to it.

Subtract this logarithm from the given one; multiply the result by 60, and divide by the tabular difference.

The quotient is the number of seconds to be added to the number of degrees and minutes already found.

To find the angle which corresponds to a given logarithmic cosine, cotangent, or cosecant.

Find the next *higher* logarithm in the tables, and take the number of degrees and minutes corresponding to it.

Subtract the given logarithm from the logarithm just found; multiply the result by 60, and divide by the tabular difference.

The quotient is the number of seconds to be added to the number of degrees and minutes already found.

To find the angle which corresponds to a given logarithmic sine, tangent, or secant, without the use of the tables.

(i) Take the difference between the two given angles.

(ii) Take the difference between their logarithms.

(iii) Subtract the logarithm of the smaller of the two angles from the given logarithm.

Then multiply this last difference (iii) by the result of (i), and divide by the logarithmic difference (ii).

The quotient is the number of seconds to be added to the smaller of the two given angles.

To find the angle which corresponds to a given logarithmic cosine, cotangent, or cosecant, without the use of the tables.

(i) Take the difference between the two given angles.

(ii) Take the difference between their logarithms.

(iii) Subtract the given logarithm from the logarithm of the smaller of the two given angles.

Then multiply this last difference (iii) by the result of (i), and divide by the logarithmic difference (ii).

The quotient is the number of seconds to be added to the smaller of the two given angles.

SOLUTION OF TRIANGLES.

In working questions involving the angles, remember that $\frac{A+B}{2}$ and $\frac{C}{2}$ are complementary; $A+B$ and C, supplementary.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

$$a = b\cos C + c\cos B.$$

$$\sin\frac{A}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}.$$

$$\cos\frac{A}{2} = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}.$$

$$\sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}.$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C.$$

$$\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1.$$

Def. A subsidiary angle is an angle introduced into an expression for the purpose of putting it into the form of a product, and thus adapting it to logarithmic computation.

To find A from the equation,

$$\sin A = n,$$

where n is nearly equal to unity.

Reduce to the form

$$\sin\left(45^\circ - \frac{A}{2}\right) = \sqrt{\left(\frac{1-n}{2}\right)}.$$

To find A from the equation,

$$\cos A = n,$$

where n is nearly equal to unity.

Reduce to the form

$$\tan\frac{A}{2} = \sqrt{\left(\frac{1-n}{1+n}\right)}.$$

PROPERTIES OF TRIANGLES.

$$\Delta = \tfrac{1}{2}bc\sin A$$

$$= \sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

$$r = (s-a)\tan\frac{A}{2}$$

$$= \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\Delta}{s}.$$

$$r_1 = s \tan\frac{A}{2}$$

$$= (s-c)\cot\frac{B}{2}$$

$$= \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\Delta}{s-a}.$$

$$R = \frac{a}{2\sin A}$$

$$= \frac{abc}{4\Delta}.$$

REGULAR POLYGONS.

$$a = 2R\sin\frac{\pi}{n}.$$

$$\text{Area} = \frac{nR^2}{2}\sin\frac{2\pi}{n}.$$

$$a = 2r\tan\frac{\pi}{n}.$$

$$\text{Area} = nr^2\tan\frac{\pi}{n}.$$

$$\text{Area} = \frac{na^2}{4}\cot\frac{\pi}{n}.$$

$$\text{Area of a circle} = \pi r^2.$$

$$\text{Area of sector of a circle} = \frac{r^2\theta}{2}.$$

De Moivre's Theorem. Whatever be the value of n, positive or negative, integral or fractional,

$$\cos n\theta + \sqrt{(-1)}\,\sin n\theta$$

is one of the values of

$$\{\cos\theta + \sqrt{(-1)}\,\sin\theta\}^n.$$

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor\underline{2}}\cos^{n-2}\theta\,\sin^2\theta$$
$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor\underline{4}}\cos^{n-4}\theta\,\sin^4\theta - \ldots.$$

$$\sin n\theta = n\cos^{n-1}\theta\,\sin\theta - \frac{n(n-1)(n-2)}{\lfloor\underline{3}}\cos^{n-3}\theta\,\sin^3\theta$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor\underline{5}}\cos^{n-5}\theta\,\sin^5\theta\ldots.$$

$$\cos\theta = 1 - \frac{\theta^2}{\lfloor\underline{2}} + \frac{\theta^4}{\lfloor\underline{4}} - \frac{\theta^6}{\lfloor\underline{6}} + \ldots.$$

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor\underline{3}} + \frac{\theta^5}{\lfloor\underline{5}} - \frac{\theta^7}{\lfloor\underline{7}} + \ldots.$$

$$\cos x = \frac{e^{x\sqrt{(-1)}} + e^{-x\sqrt{(-1)}}}{2}.$$

$$\sin x = \frac{e^{x\sqrt{(-1)}} - e^{-x\sqrt{(-1)}}}{2\sqrt{(-1)}}.$$

Gregory's series.

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots.$$

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin\{\alpha + (n-1)\beta\}$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\{\alpha+(n-1)\beta\}$$
$$= \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

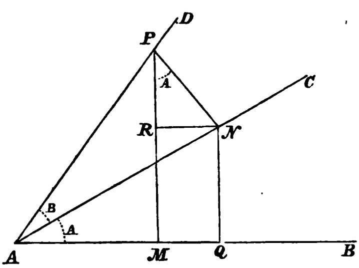
$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

$$\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots.$$

To shew that

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}.$$

Let the angle BAC be denoted by A, and the angle CAD by B; then the angle BAD will be denoted by $A+B$.



In AD take any point P.

From P draw PM perpendicular to AB, and PN perpendicular to AC.

From N draw NQ perpendicular to AB, and NR perpendicular to PM.

Then the angle $NPR = 90° - PNR = RNA = A$.

Thus $$\tan(A+B) = \frac{PM}{AM} = \frac{RM+PR}{AQ-QM}$$

$$= \frac{NQ+PR}{AQ-NR}$$

$$= \frac{\frac{NQ}{AQ} + \frac{PR}{AQ}}{1 - \frac{NR}{AQ}}$$

$$= \frac{\frac{NQ}{AQ} + \frac{PR}{PN} \cdot \frac{PN}{AN} \cdot \frac{AN}{AQ}}{1 - \frac{NR}{PN} \cdot \frac{PN}{AN} \cdot \frac{AN}{AQ}} *$$

$$= \frac{\tan A + \cos A . \tan B . \sec A}{1 - \sin A \tan B \sec A}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

To shew that

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A . \tan B}.$$

Let the angle BAC be denoted by A, and the angle CAD by B; then the angle BAD will be denoted by $(A-B)$.

In AD take any point P.

From P draw PM perpendicular to AB, and PN perpendicular to AC.

From N draw NQ perpendicular to AB, and NR perpendicular to MP produced.

Then the angle

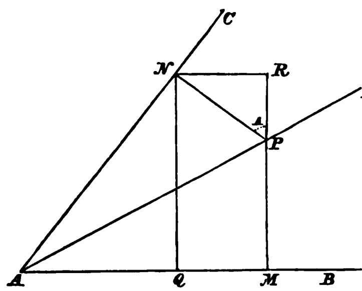
$$NPR = 90° - PNR = RNC = BAC = A.$$

---

* The student is recommended first to write $\frac{PR}{AQ}$ thus:

$$\frac{PR}{\quad} \; \frac{\quad}{AQ},$$

and then fill in PN and AN, which are the only lines connecting PR and AQ, from which we can get ratios.



Thus, $\tan(A-B) = \dfrac{PM}{AM} = \dfrac{MR-PR}{AQ+QM}$

$$= \frac{NQ-PR}{AQ+NR}$$

$$= \frac{\dfrac{NQ}{AQ} - \dfrac{PR}{AQ}}{1+\dfrac{NR}{AQ}}$$

$$= \frac{\dfrac{NQ}{AQ} - \dfrac{PR}{PN}\cdot\dfrac{PN}{AN}\cdot\dfrac{AN}{AQ}}{1+\dfrac{NR}{PN}\cdot\dfrac{PN}{AN}\cdot\dfrac{AN}{AQ}}$$

$$= \frac{\tan A - \cos A . \tan B . \sec A}{1+\sin A \tan B \sec A}$$

$$= \frac{\tan A - \tan B}{1+\tan A . \tan B}.$$

To shew that, when θ is indefinitely diminished,

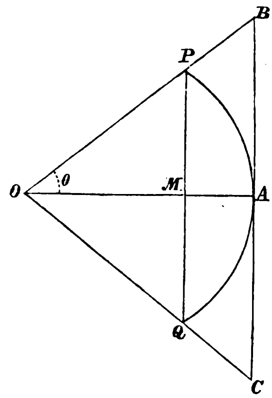
$$\sin\theta = \theta = \tan\theta.$$

Let O be the centre of a circle;

PQ a chord subtending the arc PAQ, less than a semi-circle;

OA a radius bisecting PQ at right angles at M;

BC a tangent to the circle at A, terminated by the radii OP, OQ, produced to meet it at B and C respectively.



Let θ be the circular measure of the angle AOP;

therefore $$\theta = \frac{\text{arc } PA}{OP};$$

$$\sin\theta = \frac{PM}{OP};$$

$$\tan\theta = \frac{AB}{OA} = \frac{AB}{OP}.$$

We now make two assumptions:

(i) that the chord $PQ <$ the arc PAQ, and therefore that $PM <$ the arc PA;

(ii) that the arc $PA < BA$.

Thus PM, the arc PA, and BA are in ascending order of magnitude;

therefore $\sin\theta$, θ, and $\tan\theta$ are in ascending order of magnitude;

therefore 1, $\dfrac{\theta}{\sin\theta}$, and $\dfrac{1}{\cos\theta}$ are in ascending order of magnitude;

therefore $\dfrac{\theta}{\sin\theta}$ lies in value between 1 and $\dfrac{1}{\cos\theta}$.

But $$lt_{\theta=0}\left(\frac{1}{\cos\theta}\right)=1;$$

therefore $$lt_{\theta=0}\left(\frac{\theta}{\sin\theta}\right)=1.$$

Also $$\frac{\tan\theta}{\theta}=\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta};$$

therefore $$lt_{\theta=0}\left(\frac{\tan\theta}{\theta}\right)=1.$$

Thus $\sin\theta=\theta=\tan\theta$, when θ is indefinitely diminished.

THE AMBIGUOUS CASE.

To solve a triangle having given two sides and an angle opposite to one of them.

Let a, b be the given sides,

and A the given angle.

Then $$\frac{\sin B}{\sin A}=\frac{b}{a};$$

therefore $$\sin B=\frac{b\sin A}{a}.$$

Having found B, we have

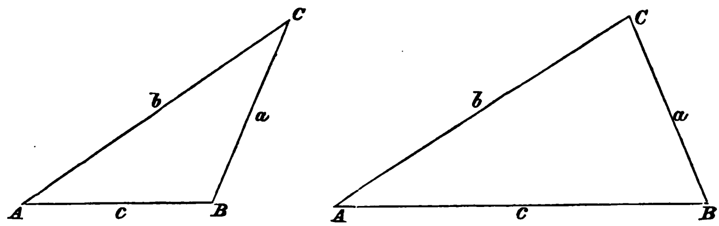
$$C=180^\circ-A-B,$$

and $$\frac{c}{a}=\frac{\sin C}{\sin A}.$$

I. If $$\frac{b\sin A}{a}<1,$$

there are two different angles that have this sine, one of them being the supplement of the other, and hence one acute, and the other obtuse.

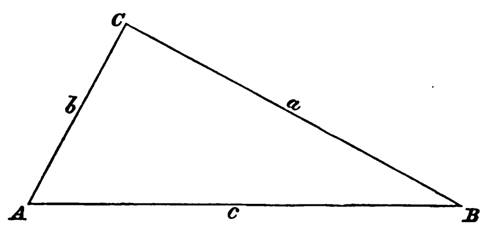
If $a<b$, then $A<B$;
therefore either value may be taken for B.



Thus we shall have two corresponding values for C, and two for c.

In this case, then, we get two triangles having the given parts.

If a is $>b$, then A is $>B$;



therefore, since we cannot have two obtuse angles in a triangle (Euclid I. 17),

B must be an acute angle.

Hence, in this case, we can only take the smaller value of B, and get one triangle with the given parts.

If $a=b$, then $A=B$. (Euclid I. 5).
Thus we get one isosceles triangle.

II. If
$$\frac{b \sin A}{a}=1,$$
then B is a right angle,
and hence we get one right-angled triangle.

III. If $$\frac{b \sin A}{a} > 1,$$

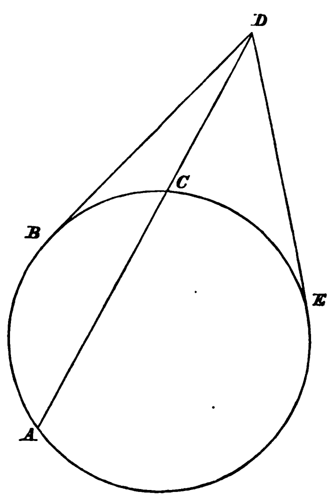
no triangle exists with the given parts, since the sine of an angle cannot be greater than unity.

The following results are frequently required in the investigation of the properties of triangles.

If from any point without a circle two straight lines be drawn touching the circle, they are equal.

This follows immediately from Euclid III. 37.

For if DB, DE be two straight lines touching a circle ABC in the points B and E, and DCA be any other line cutting the circle,



then sq. on DB = rect. $DC.DA$ = sq. on DE;

therefore $$DB = DE.$$

If $$2s = a + b + c,$$

then $$2s - 2a = b + c - a;$$

therefore $$s-a=\frac{b+c-a}{2}.$$

Similarly $$s-b=\frac{c+a-b}{2},$$

$$s-c=\frac{a+b-c}{2}.$$

Thus $$(s-a)+(s-b)+(s-c)=s,$$
$$(s-b)+(s-c)=a,$$
$$(s-c)+(s-a)=b,$$
$$(s-a)+(s-b)=c.$$

In any triangle

$$A+B+C=180^\circ \ldots \text{(Euclid I. 32)};$$

therefore $$\frac{A+B+C}{2}=90^\circ;$$

thus $\frac{A}{2}$ and $\frac{B+C}{2}$, $\frac{B}{2}$ and $\frac{C+A}{2}$, $\frac{C}{2}$ and $\frac{A+B}{2}$ are complementary.

To find the magnitude of certain lines and angles connected with triangles and their circles.

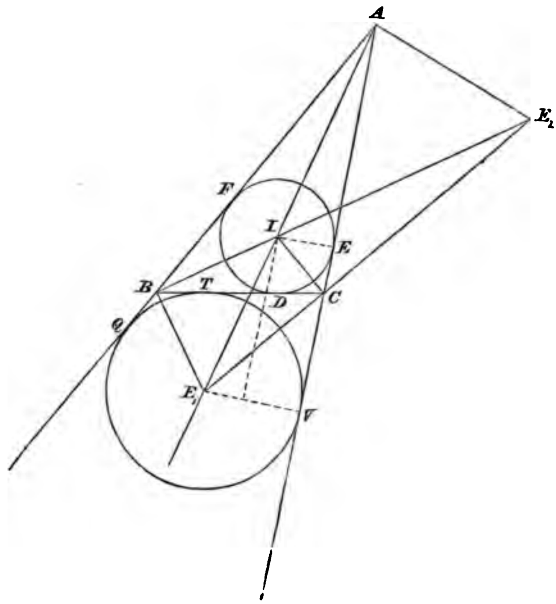
Let ABC be any triangle.

Let I be the centre, and D, E, F the points of contact of the inscribed circle; E_1 the centre, and Q, T, V the points of contact of the escribed circle with respect to the angle A.

Then

$$AF=AE, \text{ therefore } 2AF=AE+AF;$$
$$BD=BF, \text{ therefore } 2BD=BF+BD;$$
$$CD=CE, \text{ therefore } 2CD=CD+CE;$$

adding, $$2(AF+BD+CD)=2s,$$



and $\qquad BD + CD = a;$

therefore $\qquad AF + a = s;$

$$\left.\begin{array}{ll}\text{therefore} & AF \text{ or } AE = s - a; \\ \text{therefore} & BF \text{ or } BD = s - b; \\ \text{and} & CD \text{ or } CE = s - c.\end{array}\right\} \text{..............(i)}$$

Again, $\qquad AQ = AV,$

and $\qquad BQ = BT,\;$ therefore $AQ = c + BT;$

$\qquad CV = CT,\;$ therefore $AV = b + CT;$

therefore $\qquad AQ + AV = a + b + c = 2s;$

$$\left.\begin{array}{ll}\text{therefore} & AQ = AV = s; \\ \text{therefore} & BQ = BT = s - c, \\ \text{and} & CV = CT = s - b.\end{array}\right\} \text{.............. (ii).}$$

Again, $\qquad EV = EC + CV$

$$= (s - b) + (s - c)$$

$$= a \text{ .... (iii),}$$

$$AE_1 = AV \sec\frac{A}{2} = s \sec\frac{A}{2} \text{ .... (iv)},$$

$$IA = AE \sec\frac{A}{2} = (s-a) \sec\frac{A}{2} \text{ .... (v)}.$$

By subtraction we may find IE_1, or independently thus:

Draw IW parallel to AV meeting E_1V in W.

Then $$IW = EV = a,$$

$$\text{and the angle } E_1IW = \frac{A}{2};$$

therefore $$IE_1 = a \sec\frac{A}{2} \text{ .... (vi)}.$$

Similarly, $IE_2 = b \sec\frac{B}{2}$, and $IE_3 = c \sec\frac{C}{2}$.

Join E_1B, E_1C.

Then since a circle can be described round IBE_1C, the angle $IE_1C = \frac{B}{2}$ in the same segment,

and the angle $IE_1B = \frac{C}{2}$ for the same reason;

therefore the angle $BE_1C = \frac{B+C}{2}$.

Thus the triangle formed by joining the centres of the escribed circles has its angles respectively the complements of the semi-angles of the original triangle.

Using the Lemma, page 37, we obtain many neat expressions for the radii of the different circles.

Thus, for example, we have

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$= \frac{a+b+c}{\sin A + \sin B + \sin C};$$

therefore $$R = \frac{s}{\sin A + \sin B + \sin C} \text{..............(i)},$$

$$= \frac{s}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \text{............(ii)}.$$

Again,

$$r = \frac{s-a}{\cot\frac{A}{2}} = \frac{s-b}{\cot\frac{B}{2}} = \frac{s-c}{\cot\frac{C}{2}}$$

$$= \frac{s}{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}$$

$$= \frac{s}{\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}}$$

$$= s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \text{...........(iii)}.$$

From (ii) and (iii)

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2};$$

therefore $$R + r = R\left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

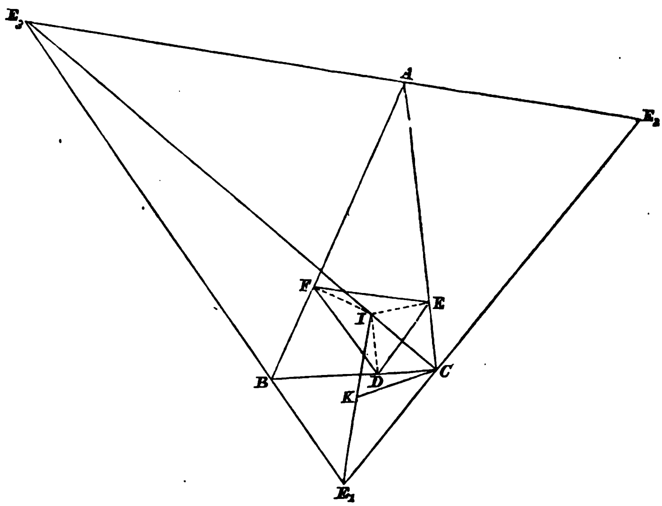
$$= R(\cos A + \cos B + \cos C).$$

ABC is a triangle; D, E, F the points of contact of the inscribed circle; E_1, E_2, E_3 the centres of the escribed circles; to compare the areas of the triangles DEF, ABC, $E_1E_2E_3$.

Since the angles IEA, IFA are right angles,

the angle $FIE = 180° - A$;

therefore $$\triangle FIE = \frac{r^2}{2}\sin A;$$



hence the whole $\triangle\, DEF = \frac{r^2}{2}(\sin A + \sin B + \sin C)$

$$= \frac{r^2 s}{2R}$$

$$= \frac{r}{2R} \cdot \triangle\, ABC;$$

thus $$\frac{\triangle\, DEF}{\triangle\, ABC} = \frac{r}{2R}.$$

Again, the radius of the circumscribing circle of the triangle $E_1E_2E_3 = 2R$;

therefore $$E_1E_2 = 4R \sin E_1E_3E_2$$

$$= 4R \cos \frac{C}{2};$$

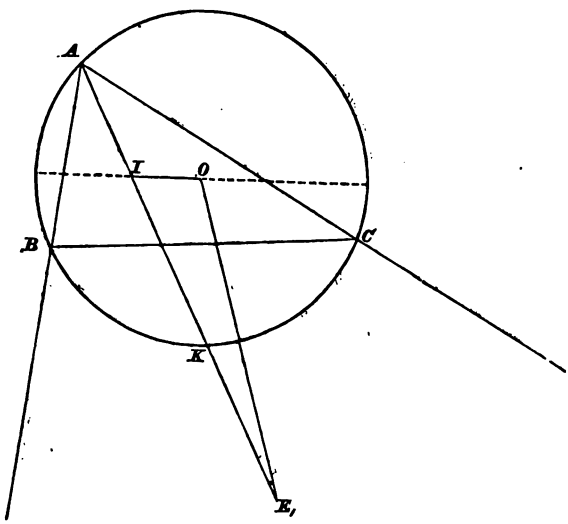
and $$CE_3 = s \sec \frac{C}{2};$$

therefore $$\triangle\, E_1E_2E_3 = \tfrac{1}{2} CE_3 . E_1E_2 = 2Rs;$$

but $$\triangle ABC = rs;$$

therefore $$\frac{\triangle ABC}{\triangle E_1E_2E_3} = \frac{r}{2R} = \frac{\triangle DEF}{\triangle ABC}.$$

To find the distances of the centre of the circumscribing circle from the centres of the inscribed and escribed circles.



Let O be the centre of the circumscribing circle, and I the centre of the inscribed circle of the triangle ABC. Produce AI to E_1 the centre of the escribed circle with respect to the angle A.

The circumscribing circle bisects IE_1 at K.

Join OI, OE_1.

Then $(R - OI)(R + OI) = AI.IK$; (Euclid III. 35),

but $$AI = \frac{r}{\sin\frac{A}{2}},$$

and $$IK = KC = 2R \sin \frac{A}{2};$$

therefore $$R^2 - OI^2 = 2R \sin \frac{A}{2} \cdot \frac{r}{\sin \frac{A}{2}} = 2Rr;$$

therefore $$OI^2 = R^2 - 2Rr.$$

Therefore $$OI = \sqrt{(R^2 - 2Rr)}.$$

Again, if T be the length of a tangent from E_1 to the circumscribing circle,

$$\begin{aligned} OE_1^2 &= R^2 + T^2 \text{ (Euclid I. 47)} \\ &= R^2 + E_1A \,.\, E_1K \text{ (Euclid III. 36)} \\ &= R^2 + \frac{r_1}{\sin \frac{A}{2}} \,.\, 2R \sin \frac{A}{2} \\ &= R^2 + 2Rr_1; \end{aligned}$$

therefore $$OE_1 = \sqrt{(R^2 + 2Rr_1)}.$$

Similarly

$$OE_2 = \sqrt{(R^2 + 2Rr_2)}, \text{ and } OE_3 = \sqrt{(R^2 + 2Rr_3)}.$$

EXAMPLES.

Using the Lemma, page 37, prove the following relations:

1. $4R \sin \frac{C}{2} \sin \frac{A-B}{2} = a - b.$

2. $r \left(\cot \frac{B}{2} - \cot \frac{A}{2}\right) = a - b.$

3. $r \left(\sin^2 \frac{A}{2} - \sin^2 \frac{B}{2}\right) = (b - a) \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$

4. $R(\sin 2A+\sin 2B+\sin 2C)=a\cos A+b\cos B+c\cos C.$

5. $2Rc=ab(\cot A+\cot B).$

6. $2R=\dfrac{a\tan\frac{A}{2}+b\tan\frac{B}{2}+c\tan\frac{C}{2}}{3-(\cos A+\cos B+\cos C)}.$

7. $\Delta\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right)=s^2.$

8. $r_1\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)=b-c.$

9. $2R(\cos A+\cos B+\cos C)=a\cot A+b\cot B+c\cot C.$

10. $2(R+r)=a\cot A+b\cot B+c\cot C.$

11. $2R(a\sin A+b\sin B+c\sin C)=a^2+b^2+c^2.$

12. $a^2+b^2+c^2-s^2=r^2\left(\cot^2\frac{A}{2}+\cot^2\frac{B}{2}+\cot^2\frac{C}{2}\right).$

13. $(a+2b)\sin A+(b+2c)\sin B+(c+2a)\sin C=\dfrac{2s^2}{R}.$

14. $2R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=s-a.$

15. $(s-a)\cot\frac{A}{2}+(s-b)\cot\frac{B}{2}+(s-c)\cot\frac{C}{2}$

$$=\frac{a^2+b^2+c^2-s^2}{r}.$$

16. $a^2-b^2=4R^2\sin C\sin(A-B).$

17. $2R(bc\sin A+ca\sin B-ab\sin C)=abc.$

18. If $\quad l\tan\frac{A}{2}=m\tan\frac{B}{2}=n\tan\frac{C}{2},$

each $=\frac{r}{s}(l+m+n).$

19. $4\Delta(\cot A+\cot B+\cot C)=a^2+b^2+c^2.$

20. $R(a\sin A+b\sin B+c\sin C)$

$$=ab\cos C+bc\cos A+ca\cos B.$$

Prove the following relations between the sides of any triangle and the radii of its circles:

21. $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$.

22. $r_1 + r_2 + r_3 - r = 4R$.

23. $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$.

24. $R = \dfrac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{4(r_1 r_2 + r_2 r_3 + r_3 r_1)}$.

25. $\dfrac{r_1 + r_2}{c} + \dfrac{r_2 + r_3}{a} + \dfrac{r_3 + r_1}{b} = \dfrac{s}{r}$.

26. $r = \dfrac{ab - r_1 r_2}{r_3} = \dfrac{bc - r_2 r_3}{r_1} = \dfrac{ca - r_3 r_1}{r_2}$.

27. $\dfrac{r_1 + r_2}{r_1 r_2} + \dfrac{r_2 + r_3}{r_2 r_3} + \dfrac{r_3 + r_1}{r_3 r_1} = \dfrac{2}{r}$.

28. $\dfrac{(r+r_1)(r+r_2)}{(b+c)(c+a)} + \dfrac{(r+r_2)(r+r_3)}{(c+a)(a+b)} + \dfrac{(r+r_3)(r+r_1)}{(a+b)(b+c)} = 1$.

29. $rr_1 s(s-a) = r_2 r_3 (s-b)(s-c)$.

30. $4Rrs = abc$.

31. If α, β, γ be the distances between the centres of the escribed circles, shew that

$$8r_1 r_2 r_3 = \alpha\beta\gamma \sin A \sin B \sin C.$$

ON STYLE.

In writing out mathematical propositions, we must observe certain important points of style.

1. A fresh line must be commenced with each new assertion.

2. A figure must occupy a space in the middle of the paper without any writing on either side.

3. Capital letters must be used to denote angles.

4. Small letters are used to denote lengths of sides.

5. Equations must occupy separate lines, and be placed in the middle of the paper.

The following symbols and abbreviations are allowed.

$\therefore$	for	*therefore.*
$\because$	,,	*since.*
$=$	,,	*is (or are) equal to.*
$\angle$	,,	*angle.*
$\perp$r	,,	*perpendicular.*
$\parallel$	,,	*parallel.*
$\parallel$gram	,,	*parallelogram.*
$\triangle$	,,	*triangle.*
$\odot$	,,	*circle.*
$\bigcirc$ce	,,	*circumference.*
(la	,,	*parabola.*
)(la	,,	*hyperbola.*
sq. on	,,	*square described on.*
sqq. on	,,	*squares described on.*
rect. AB, AC	,,	*rectangle contained by* AB, AC.

N.B. The symbol = must *not* be used for the *adjective* equal. It is a mistake to write = sides for equal sides.

FINIS.

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